



**ARULMIGU PALANIANDAVR ARTS COLLEGE FOR WOMEN**  
(Autonomous)

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**DEPARTMENT OF COMPUTER SCIENCE  
AND  
COMPUTER APPLICATION**

**LEARNING RESOURCES**

**DESIGN AND ANALYSIS OF ALGORITHM**

## Introduction to Design and analysis of algorithms

### What is an algorithm?

Algorithm is a *set of steps to complete a task.*

For example,

Task: to make a  
cup of tea.

Algorithm:

- add water and milk to the kettle,
- boilit, add tea leaves,
- Add sugar, and then serve it in cup.

### What is *Computer algorithm*?

*“a set of steps to accomplish or complete a task that is described precisely enough that acomputer can run it”.*

**Described precisely:** very difficult for a machine to know how much water, milk to be addedetc. in the above tea making algorithm.

These algorithms run on computers or computational devices. For example, GPS in oursmart phones, Google hangouts.

GPS uses *shortest path algorithm*. Online shopping uses cryptography which uses RSAalgorithm.

### Characteristics of an algorithm:-

- Must take an input.
- Must give some output(yes/no,valueetc.)
- Definiteness –each instruction is clear and unambiguous.
- Finiteness –algorithm terminates after a finite number of steps.
- Effectiveness –every instruction must be basic i.e. simple instruction

## Expectation from an algorithm

- Correctness:-
  - Correct: Algorithms must produce correct result.
  - Produce an incorrect answer: Even if it fails to give correct results all the time still there is a control on how often it gives wrong result. E.g. Rabin-Miller Preliminary Test (Used in RSA algorithm): It doesn't give correct answer all the time. 1 out of  $2^{50}$  times it gives incorrect result.
  - Approximation algorithm: Exact solution is not found, but near optimal solution can be found out. (Applied to optimization problem.)
- Less resource usage:

Algorithms should use fewer resources (time and space).

### Resource usage:

Here, the time is considered to be the primary measure of efficiency. We are also concerned with how much the respective algorithm involves the computer memory. But mostly time is the resource that is dealt with. And the actual running time depends on a variety of backgrounds: like the speed of the Computer, the language in which the algorithm is implemented, the compiler/interpreter, skill of the programmers etc.

*So, mainly the resource usage can be divided into: 1. Memory (space) 2. Time*

### Time taken by an algorithm?

- Performance measurement or A posteriori Analysis:
- Implementing the algorithm in a machine and then calculating the time taken by the system to execute the program successfully.
- Performance Evaluation or A priori Analysis. Before implementing the algorithm in a system. This is done as follows

1. How long the algorithm takes :-will be represented as a function of the size of the input.

$f(n)$ →how long it takes if 'n' is the size of input.

2. How fast the function that characterizes the running time grows with the input size.

“Rate of growth of running time”.

The algorithm with less rate of growth of running time is considered better.

### How algorithm is a technology ?

Algorithms are just like a technology. We all use latest and greatest processors but we need to run implementations of good algorithms on that computer in order to properly take benefits of our money that we spent to have the latest processor. Let's make this example more concrete by pitting a faster computer (computer A) running a sorting algorithm whose running time on  $n$  values grows like  $n^2$  against a slower computer (computer B) running a sorting algorithm whose running time grows like  $n \lg n$ . They each must sort an array of 10 million numbers. Suppose that computer A executes 10 billion instructions per second (faster than any single sequential computer at the time of this writing) and computer B executes only 10 million instructions per second, so that computer A is 1000 times faster than computer B in raw computing power. To make the difference even more dramatic, suppose that the world's craftiest programmer codes in machine language for computer A, and the resulting code requires  $2n^2$  instructions to sort  $n$  numbers. Suppose further that just an average programmer writes for computer B, using a high-level language with an inefficient compiler, with the resulting code taking  $50n \lg n$  instructions.

#### Computer A (Faster)

Running time grows like  $n^2$ .

10 billion instructions per sec.

$2n^2$  instruction.

#### Computer B(Slower)

Grows in  $n \log n$ .

10 million instruction per sec

$50 n \log n$  instruction.

$$\text{Time taken} = \frac{2 \times (10^7)^2}{10^{10}} = 20,000$$

$$\frac{50 \times 10^7 \times \log 10^7}{10^7} \approx 1163$$

It is more than 5.5hrs

it is under 20 mins.

So choosing a good algorithm (algorithm with slower rate of growth) as used by computer B affects a lot.

## Part- 2 - Growth of Functions ( Asymptotic notations)

Before going for growth of functions and asymptotic notation let us see how to analyse an algorithm.

### How to analyze an Algorithm

Let us form an algorithm for Insertion sort (which sort a sequence of numbers). The pseudo code for the algorithm is given below.

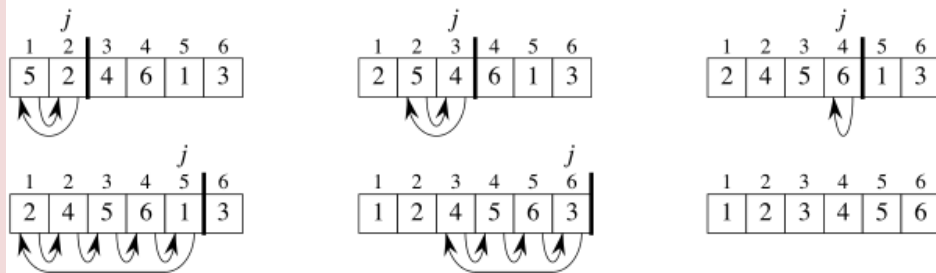
Pseudo code:

```

for j=2 to A length.....C1
key=A[j].....C2
//Insert A[j] into sorted Array A[1.....j-1]----- C3
i=j-1.....C4
while i>0 & A[j]>key-----C5
A[i+1]=A[i]-----C6
i=i-1.....C7
A[i+1]=key.....C8

```

**Example:**



Let  $C_i$  be the cost of  $i^{\text{th}}$  line. Since comment lines will not incur any cost

$C_3=0$ . Cost No. Of times Executed

$C_1n$

$C_2 n-$

1

$C_3=0 n-$

$1C_4n-1$

$C_5 \sum_{j=2}^{n-1} t_j$

$C_6 \sum_{j=2}^n (t_j - 1)$

$C_7 \sum_{j=2}^n (t_j -$

$1) C_8n-1$

Running time of the algorithm is:

$T(n)=C_1n+C_2(n-1)+0(n-1)+C_4(n-1)+C_5 (\sum_{j=2}^{n-1} t_j)+C_6(\sum_{j=2}^n t_j - )+C_7(\sum_{j=2}^n t_j - )+ C_8(n-1)$

**Best case:**

It occurs when Array is sorted.

All  $t_j$  values are 1.

$$\begin{aligned} T(n) &= C_1n + C_2(n-1) + 0(n-1) + C_4(n-1) + C_5(\sum_{j=2}^{n-1} 1) + C_6(\sum_{j=2}^n 0) + C_7(\sum_{j=2}^n 0) + C_8(n-1) \\ &= C_1n + C_2(n-1) + 0(n-1) + C_4(n-1) + C_5(n-1) + C_8(n-1) \\ &= (C_1 + C_2 + C_4 + C_5 + C_8)n - (C_2 + C_4 + C_5 + C_8) \end{aligned}$$

- Which is of the form  $an + b$ .
- Linear function of  $n$ . So, linear growth.

### Worst case:

It occurs when Array is reverse sorted, and  $t_j = j$

$$\begin{aligned} T(n) &= C_1n + C_2(n-1) + 0(n-1) + C_4(n-1) + C_5(\sum_{j=2}^{n-1} j) + C_6(\sum_{j=1}^n j) + C_7(\sum_{j=1}^n j) + C_8(n-1) \\ &= C_1n + C_2(n-1) + C_4(n-1) + C_5(\frac{n(n-1)}{2} - 1) + C_6(\sum_{j=1}^n j) + C_7(\sum_{j=1}^n j) + C_8(n-1) \end{aligned}$$

Which represents  $an^2 + bn + c$

Quadratic function. So in worst case insertion set grows in

$n^2$ . Why we concentrate on worst-case running time?

- The worst-case running time gives a guaranteed upper bound on the running time for any input.
- For some algorithms, the worst case occurs often. For example, when searching, the worst case often occurs when the item being searched for is not present, and searches for absent items may be frequent.
- Why not analyze the average case? Because it's often about as bad as the worst case.

Order of growth:

It is described by the highest degree term of the formula for running time. (Drop lower-order terms. Ignore the constant coefficient in the leading term.)

Example: We found out that for insertion sort the worst-case running time is of the form  $an^2 + bn + c$ .

Drop lower-order terms. What remains is  $an^2$ . Ignore constant coefficient. It results in  $n^2$ . But we cannot say that the worst-case running time  $T(n)$  equals  $n^2$ . Rather It grows like  $n^2$ . But it doesn't equal  $n^2$ . We say that the running time is  $\Theta(n^2)$  to capture the notion that the order of growth is  $n^2$ .

We usually consider one algorithm to be more efficient than another if its worst-case running time has a smaller order of growth.

### Asymptotic notation

- It is a way to describe the characteristics of a function in the limit.
- It describes the rate of growth of functions.
- Focus on what's important by abstracting away low-order terms and constant factors.
- It is a way to compare "sizes" of

functions:  $O \approx \leq$

$\Omega \approx$

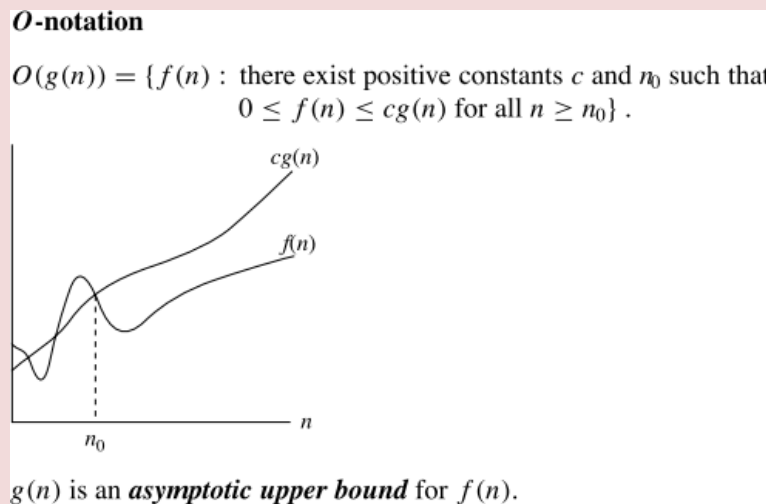
$\geq \Theta$

$\approx = o$

$\approx <$

$\omega \approx$

$>$





**Example:**  $2n^2 = O(n^3)$ , with  $c = 1$  and  $n_0 = 2$ .

Examples of functions in  $O(n^2)$ :

$$n^2$$

$$n^2 + n$$

$$n^2 + 1000n$$

$$1000n^2 + 1000n$$

Also,

$$n$$

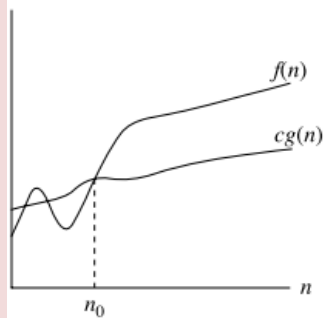
$$n/1000$$

$$n^{1.99999}$$

$$n^2 / \lg \lg \lg n$$

## $\Omega$ -notation

$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0\}$ .



$g(n)$  is an *asymptotic lower bound* for  $f(n)$ .

**Example:**  $\sqrt{n} = \Omega(\lg n)$ , with  $c = 1$  and  $n_0 = 16$ .

Examples of functions in  $\Omega(n^2)$ :

$$n^2$$

$$n^2 + n$$

$$n^2 - n$$

$$1000n^2 + 1000n$$

$$1000n^2 - 1000n$$

Also,

$$n^3$$

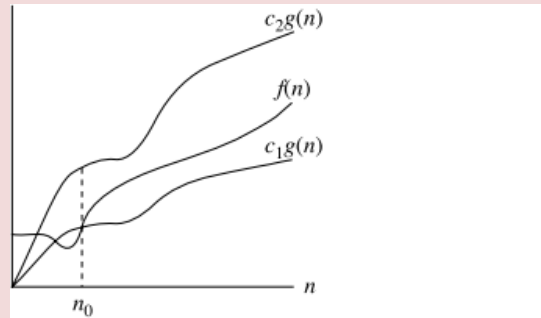
$$n^{2.00001}$$

$$n^2 \lg \lg \lg n$$

$$2^{2^n}$$

### **$\Theta$ -notation**

$\Theta(g(n)) = \{f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \text{ for all } n \geq n_0\}$ .



$g(n)$  is an *asymptotically tight bound* for  $f(n)$ .

Example:  $n^2/2 - 2n = \Theta(n^2)$ , with  $c_1 = 1/4$ ,  $c_2 = 1/2$ , and  $n_0 = 8$ .

### **$o$ -notation**

$o(g(n)) = \{f(n) : \text{for all constants } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0\}$ .

Another view, probably easier to use:  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$ .

$$n^{1.9999} = o(n^2)$$

$$n^2 / \lg n = o(n^2)$$

$$n^2 \neq o(n^2) \text{ (just like } 2 \neq 2)$$

$$n^2 / 1000 \neq o(n^2)$$

### **$\omega$ -notation**

$\omega(g(n)) = \{f(n) : \text{for all constants } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \leq cg(n) < f(n) \text{ for all } n \geq n_0\}$ .

Another view, again, probably easier to use:  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$ .

$$n^{2.0001} = \omega(n^2)$$

$$n^2 \lg n = \omega(n^2)$$

$$n^2 \neq \omega(n^2)$$

## **Recurrences, Solution of Recurrences by substitution, Recursion Tree and Master Method**

Recursion is a particularly powerful kind of reduction, which can be described loosely as follows:

- If the given instance of the problem is small or simple enough, just solve it.
- Otherwise, reduce the problem to one or more simpler instances of the same problem.

Recursion is generally expressed in terms of recurrences. In other words, when an algorithm calls to itself, we can often describe its running time by a **recurrence equation** which describes the overall running time of a problem of size  $n$  in terms of the running time on smaller inputs.

E.g. the worst case running time  $T(n)$  of the merge sort procedure by recurrence can be expressed as

$$T(n) = \Theta(1) ; \quad \text{if } n=1$$
$$2T(n/2) + \Theta(n) ; \text{if } n>1$$

whose solution can be found as  $T(n) = \Theta(n \log n)$

There are various techniques to solve recurrences.

### **1. SUBSTITUTION METHOD:**

The substitution method comprises of 3 steps

- i. Guess the form of the solution
- ii. Verify by induction
- iii. Solve for constants

We substitute the guessed solution for the function when applying the inductive hypothesis to smaller values. Hence the name “substitution method”. This method is powerful, but we must be able to guess the form of the answer in order to apply it.

E.g. recurrence equation:  $T(n) = 4T(n/2) + n$

step 1: guess the form of solution

$$T(n) = 4T(n/2)$$

$$\Rightarrow F(n) = 4f(n/2)$$

$$\Rightarrow F(2n) = 4f(n)$$

$$\Rightarrow F(n) = n^2$$

So,  $T(n)$  is order of  $n^2$

Guess  $T(n) = O(n^3)$

Step 2: verify the induction

Assume  $T(k) \leq ck^3$

$$T(n) = 4T(n/2) + n$$

$$\leq 4c(n/2)^3 + n$$

$$\leq cn^3/2 + n$$

$$\leq cn^3 - (cn^3/2 - n)$$

$T(n) \leq cn^3$  as  $(cn^3/2 - n)$  is always

positive. So what we assumed was true.

$$\Rightarrow T(n) = O(n^3)$$

Step 3: solve for

constants  $Cn^3/2 -$

$$n >= 0$$

$$\Rightarrow n >= 1$$

$$\Rightarrow c >= 2$$

Now suppose we guess that  $T(n) = O(n^2)$  which is tight upper

bound. Assume,  $T(k) \leq ck^2$

so, we should prove that  $T(n) \leq cn^2$

$$T(n) = 4T(n/2) + n$$

$$\Rightarrow 4c(n/2)^2 + n$$

$$\Rightarrow cn^2 + n$$

So,  $T(n)$  will never be less than  $cn^2$ . But if we will take the assumption of  $T(k)=c_1 k^2-c_2k$ , then we can find that  $T(n) = O(n^2)$

## 2. BY ITERATIVE METHOD:

e.g.  $T(n)=2T(n/2)+n$

$\Rightarrow 2[2T(n/4) + n/2 ]+n$

$\Rightarrow 2^2T(n/4)+n+n$

$\Rightarrow 2^2[2T(n/8)+ n/4]+2n$

$\Rightarrow 2^3T(n/2^3) +3n$

After  $k$  iterations , $T(n)=2^kT(n/2^k)+kn$  ----- (1)

Sub problem size is 1 after  $n/2^k=1 \Rightarrow k=\log n$  So,

after  $\log n$  iterations ,the sub-problem size will be 1.

So, when  $k=\log n$  is put in equation 1

$T(n)=nT(1)+n\log n$

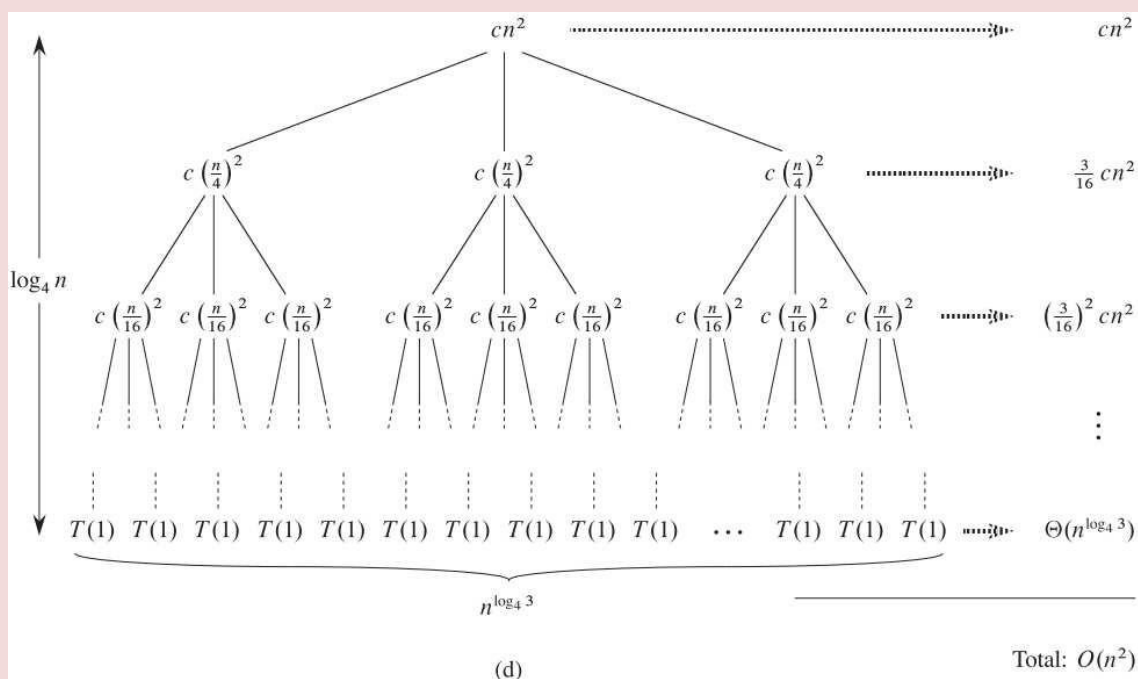
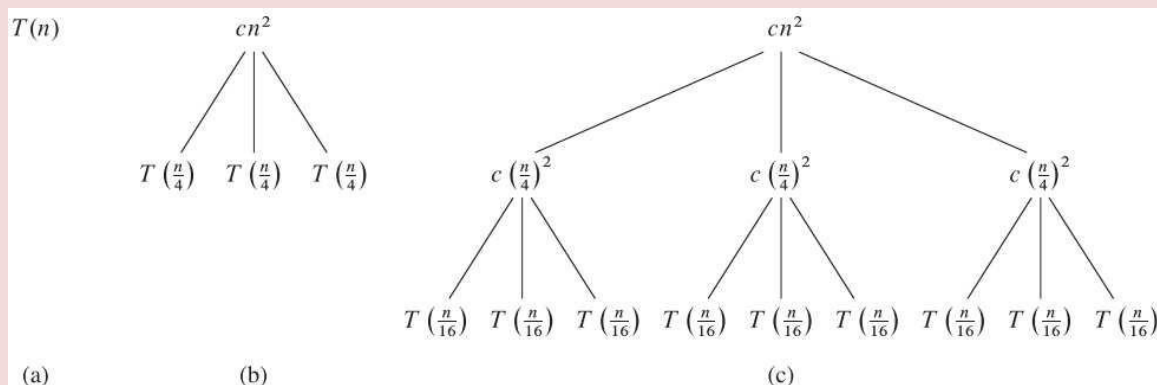
$\Rightarrow nc+n\log n$  ( say  $c=T(1)$ )

$\Rightarrow O(n\log n)$

## 3. BY RECURSSION TREE METHOD:

In a recursion tree ,each node represents the cost of a single sub-problem somewhere in the set of recursive problems invocations .we sum the cost within each level of the tree to obtain a set of per level cost, and then we sum all the per level cost to determine the total cost of all levels of recursion .

Constructing a recursion tree for the recurrence  $T(n)=3T(n/4)+cn^2$



Constructing a recursion tree for the recurrence  $T(n) = 3T(n/4) + cn^2$ . Part (a) shows  $T(n)$ , which progressively expands in (b)–(d) to form the recursion tree. The fully expanded tree in part (d) has height  $\log_4 n$  (it has  $\log_4 n + 1$  levels).

Sub problem size at depth  $i = n/4^i$

Sub problem size is 1 when  $n/4^i = 1 \Rightarrow i = \log_4 n$

So, no. of levels  $= 1 + \log_4 n$

Cost of each level  $= (\text{no. of nodes}) \times (\text{cost of each node})$

No. Of nodes at depth  $i=3^i$

Cost of each node at depth  $i=c(n/4^i)^2$

Cost of each level at depth  $i=3^i c(n/4^i)^2 =$

$$(3/16)^i cn^2 T(n) = \sum_{i=0}^{\log_4 n} cn^2 (3/16)^i$$

$$T(n) = \sum_{i=0}^{\log_4 n - 1} cn^2 (3/16)^i + \text{cost of last level}$$

Cost of nodes in last level  $= 3^i T(1)$

$$\Rightarrow c 3^{\log_4 n} \text{ (at last level } i = \log_4 n)$$

$$\Rightarrow cn^{\log_4 3}$$

$$T(n) = \sum_{i=0}^{\log_4 n - 1} cn^2 (3/16)^i + cn^{\log_4 3}$$

$$\leq cn^2 \sum_{i=0}^{\infty} (3/16)^i + cn^{\log_4 3}$$

$$\Rightarrow \leq cn^{2 * (16/13)} + cn^{\log_4 3} \Rightarrow T(n) = O(n^2)$$

#### 4. BY MASTER METHOD:

The master method solves recurrences of the form

$$T(n) = aT(n/b) + f(n)$$

where  $a \geq 1$  and  $b > 1$  are constants and  $f(n)$  is an asymptotically positive

function. To use the master method, we have to remember 3 cases:

1. If  $f(n) = O(n^{\log_b a - \epsilon})$  for some constants  $\epsilon > 0$ , then  $T(n) = \Theta(n^{\log_b a})$
2. If  $f(n) = \Theta(n^{\log_b a})$  then  $T(n) = \Theta(n^{\log_b a} \log n)$
3. If  $f(n) = \Omega(n^{\log_b a + \epsilon})$  for some constant  $\epsilon > 0$ , and if  $a * f(n/b) \leq c * f(n)$  for some constant  $c < 1$  and all sufficiently large  $n$ , then  $T(n) = \Theta(f(n))$

e.g.  $T(n)=2T(n/2)+n\log n$

ans:  $a=2$   $b=2$

$f(n)=n\log n$

using 2<sup>nd</sup> formula

$f(n)=\Theta(n^{\log_2}$

$^2\log^k n)$

$\Rightarrow\Theta(n^1 \log^k n)=n\log n$

$\Rightarrow K=1$

$T(n)=\Theta(n^{\log_2 2} \log^1 n)$

$\Rightarrow\Theta(n\log^2 n)$

## Design and analysis of Divide and Conquer Algorithms

### DIVIDE AND CONQUER ALGORITHM

- In this approach ,we solve a problem recursively by applying 3 steps
  1. **DIVIDE**-break the problem into several sub problems of smaller size.
  2. **CONQUER**-solve the problem recursively.
  3. **COMBINE**-combine these solutions to create a solution to the original problem.

### CONTROL ABSTRACTION FOR DIVIDE AND CONQUER ALGORITHM

Algorithm D and C (P)

```
{
    if small(P)
        then return S(P)
    els
    e
        { divide P into smaller instances  $P_1, P_2.. P_k$ 
          Apply D and C to each sub problem
          Return combine (D and C( $P_1$ )+ D and C( $P_2$ )+ ... +D and C( $P_k$ )
        }
}
```



Let a recurrence relation is expressed as

$T(n)=\Theta(1)$ ,

if  $n \leq C$

$aT(n/b) + D(n) + C(n)$  ,otherwise

then  $n$ =input size  $a$ =no. Of sub-problems  $n/b$ = input size of the sub-problems

### **Worst case analysis of merge sort, quick sort**

#### **Merge sort**

It is one of the well-known divide-and-conquer algorithm. This is a simple and very efficient algorithm for sorting a list of numbers.

We are given a sequence of  $n$  numbers which we will assume is stored in an array  $A [1..n]$ . The objective is to output a permutation of this sequence, sorted in increasing order. This is normally done by permuting the elements within the array  $A$ .

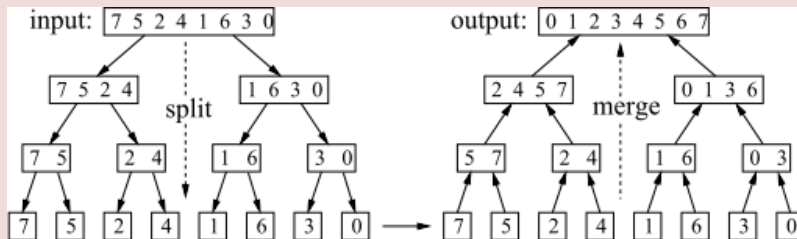
How can we apply divide-and-conquer to sorting? Here are the major elements of the Merge Sort algorithm.

Divide: Split  $A$  down the middle into two sub-sequences, each of size roughly  $n/2$  . Conquer: Sort each subsequence (by calling Merge Sort recursively on each).

Combine: Merge the two sorted sub-sequences into a single sorted list.

The dividing process ends when we have split the sub-sequences down to a single item. A sequence of length one is trivially sorted. The key operation where all the work is done is in the combine stage, which merges together two sorted lists into a single sorted list. It turns out that the merging process is quite easy to implement.

The following figure gives a high-level view of the algorithm. The “divide” phase is shown on the left. It works top-down splitting up the list into smaller sub lists. The “conquer and combine” phases are shown on the right. They work bottom-up, merging sorted lists together into larger sorted lists.



Merge Sort

Designing the Merge Sort algorithm top-down. We’ll assume that the procedure that merges two sorted list is available to us. We’ll implement it later. Because the algorithm is called recursively on sub lists, in addition to passing in the array itself, we will pass in two indices, which indicate the first and last indices of the sub array that we are to sort. The call Merge Sort(A, p, r) will sort the sub-array [ p..r ] and return the sorted result in the same sub array.

Here is the overview. If  $r = p$ , then this means that there is only one element to sort, and we may return immediately. Otherwise (if  $p < r$ ) there are at least two elements, and we will invoke the divide-and-conquer. We find the index  $q$ , midway between  $p$  and  $r$ , namely  $q = (p + r) / 2$  (rounded down to the nearest integer). Then we split the array into sub arrays  $A [ p..q ]$  and  $A [ q + 1 ..r ]$ . Call Merge Sort recursively to sort each sub array. Finally, we invoke a procedure (which we have yet to write) which merges these two sub arrays into a single sorted array.

```
Merge Sort(array A, int p, int r) {
    if (p < r) { // we have at least 2 items
        q = (p + r)/2
        Merge Sort(A, p, q) // sort A[p..q]
        Merge Sort(A, q+1, r) // sort A[q+1..r]
```

```

Merge(A, p, q, r)                                // merge everything together
}

```

Merging: All that is left is to describe the procedure that merges two sorted lists. Merge(A, p, q, r) assumes that the left sub array, A [ p..q ] , and the right sub array, A [ q + 1 ..r ] , have already been sorted. We merge these two sub arrays by copying the elements to a temporary working array called B. For convenience, we will assume that the array B has the same index range A, that is, B [ p..r ] . We have two indices i and j, that point to the current elements of each sub array. We move the smaller element into the next position of B (indicated by index k) and then increment the corresponding index (either i or j). When we run out of elements in one array, then we just copy the rest of the other array into B. Finally, we copy the entire contents of B back into A.

```

Merge(array A, int p, int q, int r) {            // merges A[p..q] with
    A[q+1..r] array B[p..r]
    i = k = p                                    //initialize pointers
    j = q+1
    while (i <= q and j <= r) {                 // while both subarrays are nonempty
        if (A[i] <= A[j]) B[k++] = A[i++]      // copy from left subarray
        else B[k++] = A[j++]                  // copy from right subarray
    }
    while (i <= q) B[k++] = A[i++]             // copy any leftover to
    Bwhile (j <= r) B[k++] = A[j++]
    for i = p to r do A[i] = B[i]              // copy B back to A
}

```

Analysis: What remains is to analyze the running time of Merge Sort. First let us consider the running time of the procedure Merge(A, p, q, r). Let  $n = r - p + 1$  denote the total length of both the left and right sub arrays. What is the running time of Merge as a function of n? The algorithm contains four loops (none nested in the other). It is easy to see that each loop can be executed at most n times. (If you are a bit more careful you can actually see that all the while-loops

Together can only be executed  $n$  times in total, because each execution copies one new element to the array  $B$ , and  $B$  only has space for  $n$  elements.) Thus the running time to Merge  $n$  items is  $\Theta(n)$ . Let us write this without the asymptotic notation; simply as  $n$ . (We'll see later why we do this.)

Now, how do we describe the running time of the entire Merge Sort algorithm? We will do this through the use of a recurrence, that is, a function that is defined recursively in terms of itself. To avoid circularity, the recurrence for a given value of  $n$  is defined in terms of values that are strictly smaller than  $n$ . Finally, a recurrence has some basis values (e.g. for  $n = 1$ ), which are defined explicitly.

Let's see how to apply this to Merge Sort. Let  $T(n)$  denote the worst case running time of Merge Sort on an array of length  $n$ . For concreteness we could count whatever we like: number of lines of pseudo code, number of comparisons, number of array accesses, since these will only differ by a constant factor. Since all of the real work is done in the Merge procedure, we will count the total time spent in the Merge procedure.

First observe that if we call Merge Sort with a list containing a single element, then the running time is a constant. Since we are ignoring constant factors, we can just write  $T(n) = 1$ . When we call Merge Sort with a list of length  $n > 1$ , e.g. Merge( $A, p, r$ ), where  $r - p + 1 = n$ , the algorithm first computes  $q = (p + r) / 2$ . The sub array  $A[p..q]$ , which contains  $q - p + 1$  elements. You can verify that is of size  $n/2$ . Thus the remaining sub array  $A[q + 1..r]$  has  $n/2$  elements in it. How long does it take to sort the left sub array? We do not know this, but because  $n/2 < n$  for  $n > 1$ , we can express this as  $T(n/2)$ . Similarly, we can express the time that it takes to sort the right sub array as  $T(n/2)$ .

Finally, to merge both sorted lists takes  $n$  time, by the comments made above. In conclusion we have

$$T(n) = 1 \text{ if } n = 1,$$

$$2T(n/2) + n \text{ otherwise.}$$

Solving the above recurrence we can see that merge sort has a time complexity of  $\Theta(n \log n)$ .

## QUICKSORT

- Worst-case running time:  $O(n^2)$ .
- Expected running time:  $O(n \lg n)$ .
- Sorts in place.

### **Description of quick sort**

Quick sort is based on the three-step process of divide-and-conquer.

- To sort the subarray  $A[p \dots r]$ :

**Divide:** Partition  $A[p \dots r]$ , into two (possibly empty) subarrays  $A[p \dots q - 1]$  and

$A[q + 1 \dots r]$ , such that each element in the first subarray  $A[p \dots q - 1]$  is  $\leq A[q]$  and  $A[q]$  is  $\leq$  each element in the second subarray  $A[q + 1 \dots r]$ .

**Conquer:** Sort the two subarrays by recursive calls to QUICKSORT.

**Combine:** No work is needed to combine the subarrays, because they are sorted in place.

- Perform the divide step by a procedure PARTITION, which returns the index  $q$  that marks the position separating the subarrays.

QUICKSORT ( $A, p, r$ )

**if**  $p < r$

**then**  $q \leftarrow$  PARTITION( $A, p, r$ )

QUICKSORT ( $A, p, q -$

$1$ ) QUICKSORT ( $A, q +$

$1, r$ )

Initial call is QUICKSORT ( $A, 1, n$ )

### **Partitioning**

Partition subarray  $A[p \dots r]$  by the following procedure:

PARTITION ( $A, p, r$ )

$x \leftarrow A[r]$

$i \leftarrow p - 1$

**for**  $j \leftarrow p$  **to**  $r - 1$

**do if**  $A[j] \leq x$

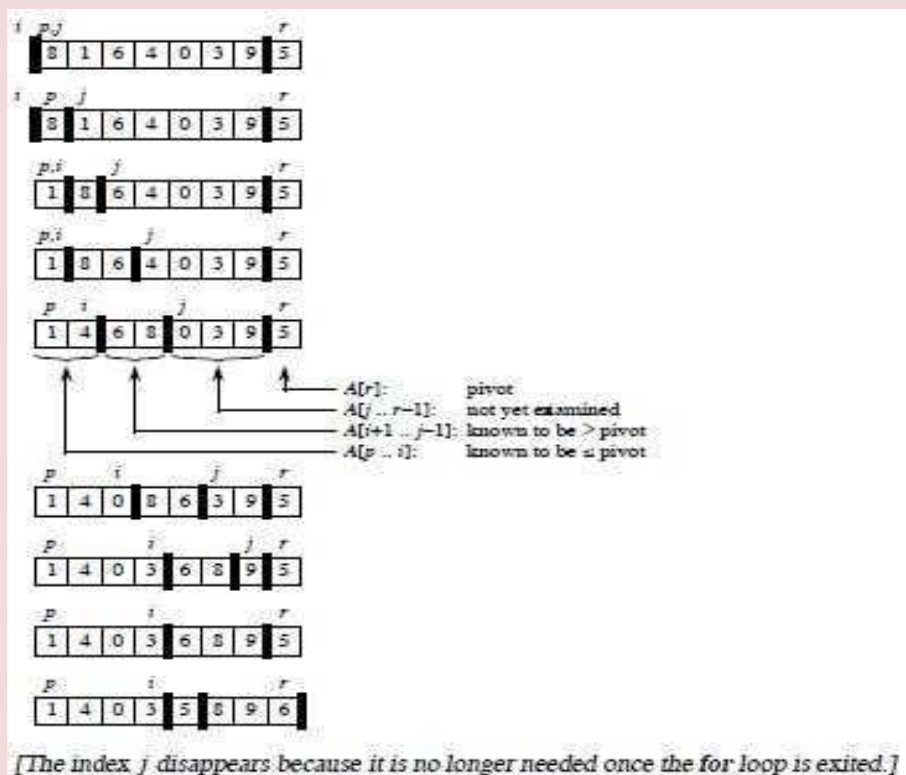
**then**  $i \leftarrow i + 1$

exchange  $A[i] \leftrightarrow A[j]$

exchange  $A[i + 1] \leftrightarrow A[r]$

return  $i + 1$

- PARTITION always selects the last element  $A[r]$  in the subarray  $A[p \dots r]$  as the *pivot* the element around which to partition.
- As the procedure executes, the array is partitioned into four regions, some of which may be empty:



### Performance of Quick sort

The running time of Quick sort depends on the partitioning of the sub arrays:

- If the sub arrays are balanced, then Quick sort can run as fast as merge sort.
- If they are unbalanced, then Quick sort can run as slowly as insertion sort.

### Worst case

- Occurs when the sub arrays are completely unbalanced.
- Have 0 elements in one sub array and  $n - 1$  elements in the other sub array.

- Get the recurrence

$$T(n) = T(n-1) + T(0) + \Theta(n)$$

$$= T(n-1) + \Theta(n)$$

$$= O(n^2).$$

- Same running time as insertion sort.
- In fact, the worst-case running time occurs when Quick sort takes a sorted array as input, but insertion sort runs in  $O(n)$  time in this case.

### Best case

- Occurs when the sub arrays are completely balanced every time.
- Each sub array has  $\leq n/2$  elements.
- Get the recurrence

$$T(n) = 2T(n/2) + \Theta(n) = O(n \lg n).$$

### Balanced partitioning

- QuickSort's average running time is much closer to the best case than to the worst case.
- Imagine that PARTITION always produces a 9-to-1 split.

- Get the recurrence

$$T(n) \leq T(9n/10) + T(n/10) + \Theta(n) = O(n \lg n).$$

- Intuition: look at the recursion tree.
- It's like the one for  $T(n) = T(n/3) + T(2n/3) + O(n)$ .
- Except that here the constants are different; we get  $\log_{10} n$  full levels and  $\log_{10/9} n$  levels that are nonempty.
- As long as it's a constant, the base of the log doesn't matter in asymptotic notation.
- Any split of constant proportionality will yield a recursion tree of depth  $O(\lg n)$ .

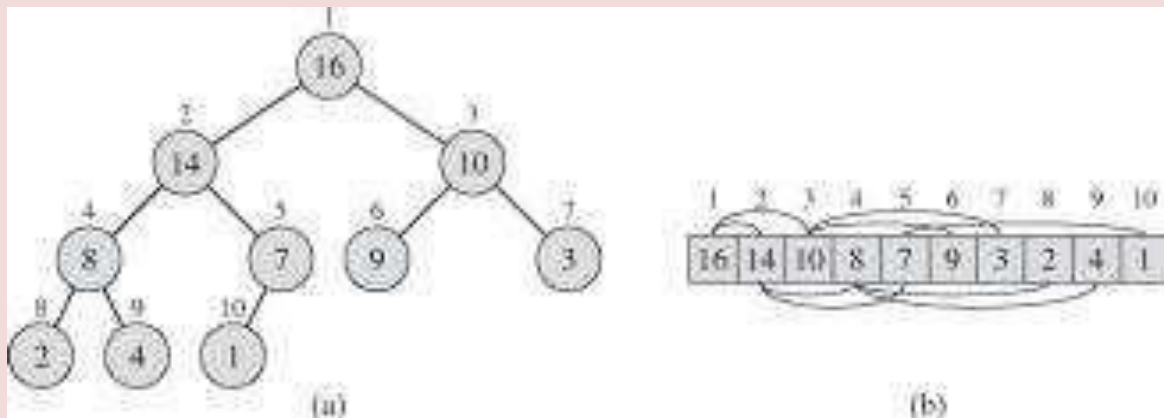
## HEAPS AND HEAP SORT

### HEAPSORT

- Inplace algorithm
- Running Time:  $O(n \log n)$
- Complete Binary Tree

The (*binary*) *heap* data structure is an array object that we can view as a nearly complete binary tree. Each node of the tree corresponds to an element of the array. The tree is completely filled on all levels except possibly the lowest, which is filled from the left up to a point.

The root of the tree is  $A[1]$ , and given the index  $i$  of a node, we can easily compute the indices of its parent, left child, and right child:



- $\text{PARENT}(i) \Rightarrow \text{return } [i/2]$
- $\text{LEFT}(i) \Rightarrow \text{return } 2i$
- $\text{RIGHT}(i) \Rightarrow \text{return } 2i+1$

On most computers, the LEFT procedure can compute  $2i$  in one instruction by simply shifting the binary representation of  $i$  left by one bit position.

Similarly, the RIGHT procedure can quickly compute  $2i + 1$  by shifting the binary representation of  $i$  left by one bit position and then adding in a 1 as the low-order bit.



The PARENT procedure can compute  $\lfloor i/2 \rfloor$  by shifting  $i$  right one bit position. Good implementations of heap sort often implement these procedures as "macros" or "inline" procedures.

There are two kinds of binary heaps: **max-heaps** and **min-heaps**.

- In a *max-heap*, the *max-heap property* is that for every node  $i$  other than the root,  $A[\text{PARENT}(i)] \geq A[i]$ , that is, the value of a node is at most the value of its parent. Thus, the largest element in a max-heap is stored at the root, and the subtree rooted at a node contains values no larger than that contained at the node itself.
- A *min-heap* is organized in the opposite way; the *min-heap property* is that for every node  $i$  other than the root,  $A[\text{PARENT}(i)] \leq A[i]$ ,

The smallest element in a min-heap is at the root.

- ✓ The height of a node in a heap is the number of edges on the longest simple downward path from the node to a leaf and
- ✓ The height of the heap is the height of its root.
- ✓ Height of a heap of  $n$  elements which is based on a complete binary tree is  $O(\log n)$ .

### Maintaining the heap property

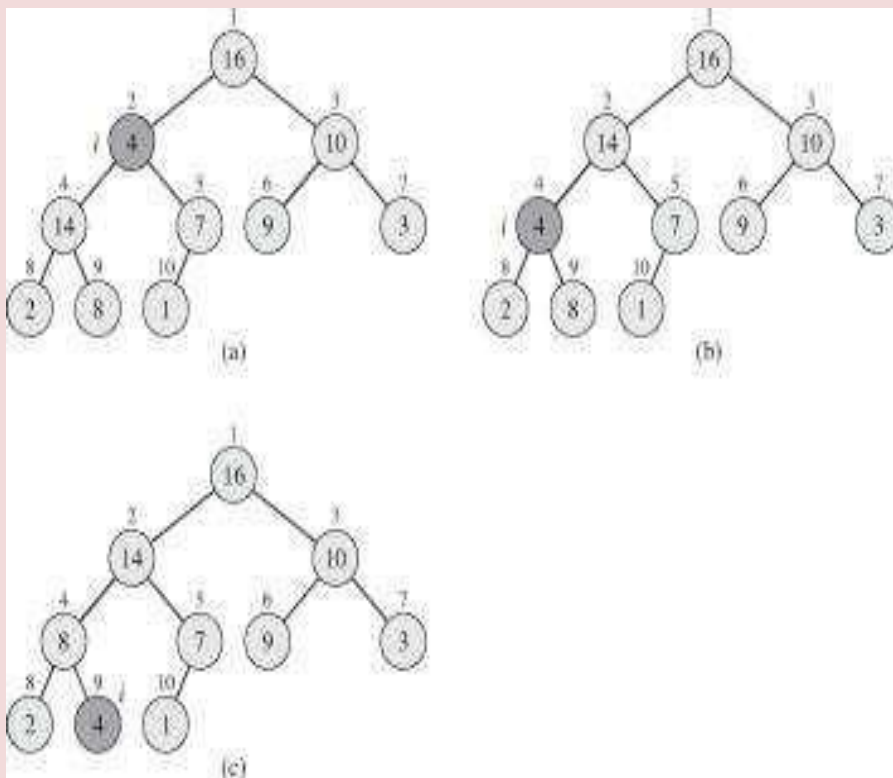
MAX-HEAPIFY lets the value at  $A[i]$  "float down" in the max-heap so that the sub tree rooted at index  $i$  obeys the max-heap property.

MAX-HEAPIFY( $A, i$ )

1.  $l \leftarrow \text{LEFT}(i)$
2.  $r \leftarrow \text{RIGHT}(i)$
3. if  $A[l] > A[i]$
4.     largest  $\leftarrow l$
5. if  $A[r] > A[\text{largest}]$
6.     largest  $\leftarrow r$
7. if largest  $\neq i$
8.     Then exchange  $A[i]$   $A[\text{largest}]$

### 9. MAX-HEAPIFY(A,largest)

At each step, the largest of the elements  $A[i]$ ,  $A[\text{LEFT}(i)]$ , and  $A[\text{RIGHT}(i)]$  is determined, and its index is stored in *largest*. If  $A[i]$  is largest, then the sub tree rooted at node  $i$  is already a max-heap and the procedure terminates. Otherwise, one of the two children has the largest element, and  $A[i]$  is swapped with  $A[\text{largest}]$ , which causes node  $i$  and its children to satisfy the max-heap property. The node indexed by *largest*, however, now has the original value  $A[i]$ , and thus the sub tree rooted at *largest* might violate the max-heap property. Consequently, we call MAX-HEAPIFY recursively on that sub tree.



**Figure:** The action of MAX-HEAPIFY (A, 2), where *heap-size* = 10. (a) The initial configuration, with  $A[2]$  at node  $i = 2$  violating the max-heap property since it is not larger than both children. The max-heap property is restored for node 2 in (b) by exchanging  $A[2]$  with  $A[4]$ , which destroys the max-heap property for node 4. The recursive call MAX-HEAPIFY (A,4)

now has  $i = 4$ . After swapping  $A[4]$  with  $A[9]$ , as shown in (c), node 4 is fixed up, and the recursive call  $\text{MAX-HEAPIFY}(A, 9)$  yields no further change to the data structure.

The running time of  $\text{MAX-HEAPIFY}$  by the recurrence can be

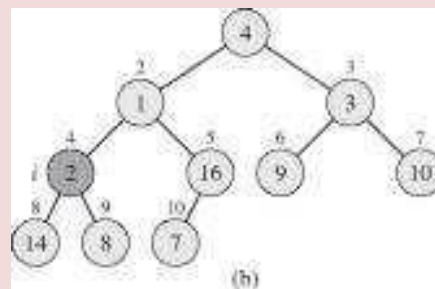
$$\text{described as } T(n) \leq T(2n/3) + O(1)$$

The solution to this recurrence is  $T(n) = O(\log n)$

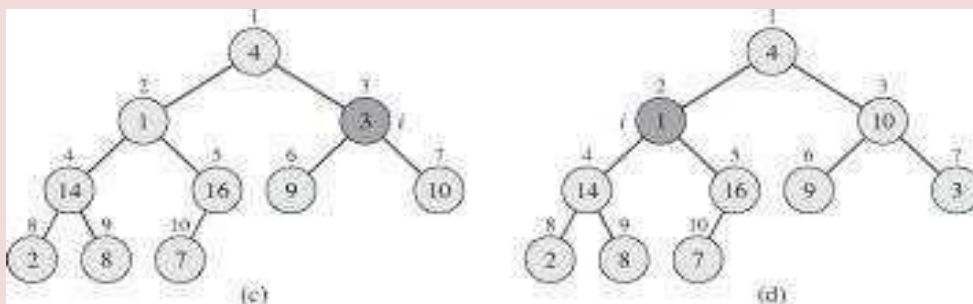
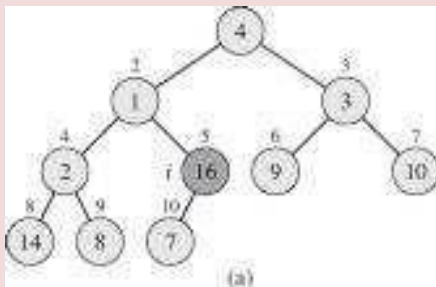
### Building a heap

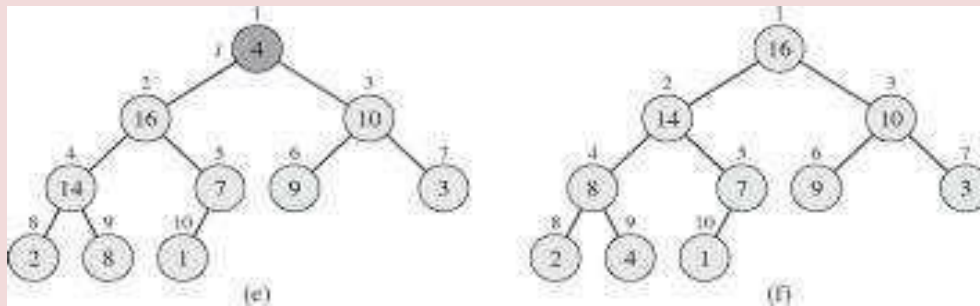
Build-Max-Heap(A)

1. for  $i \leftarrow \lfloor n/2 \rfloor$  to 1
2. do  $\text{MAX-HEAPIFY}(A, i)$



4 1 3 2 1 9 1 1 8 7





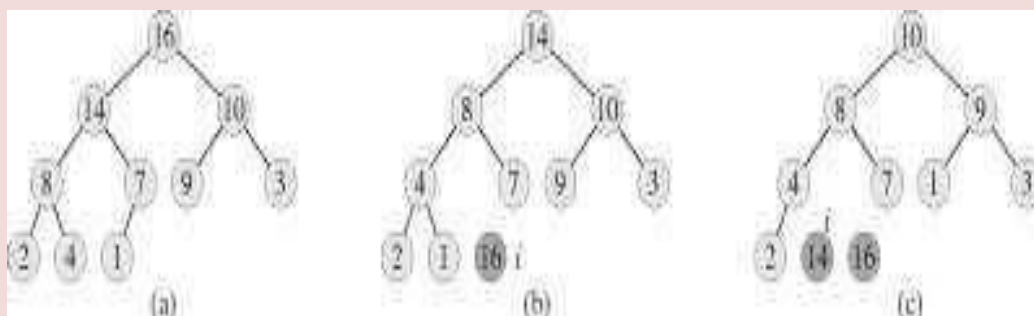
We can derive a tighter bound by observing that the time for MAX-HEAPIFY to run at a node varies with the height of the node in the tree, and the heights of most nodes are small. Our tighter analysis relies on the properties that an  $n$ -element heap has height  $\lceil \log n \rceil$  and at most  $\lceil n/2^{h+1} \rceil$  nodes of any height  $h$ .

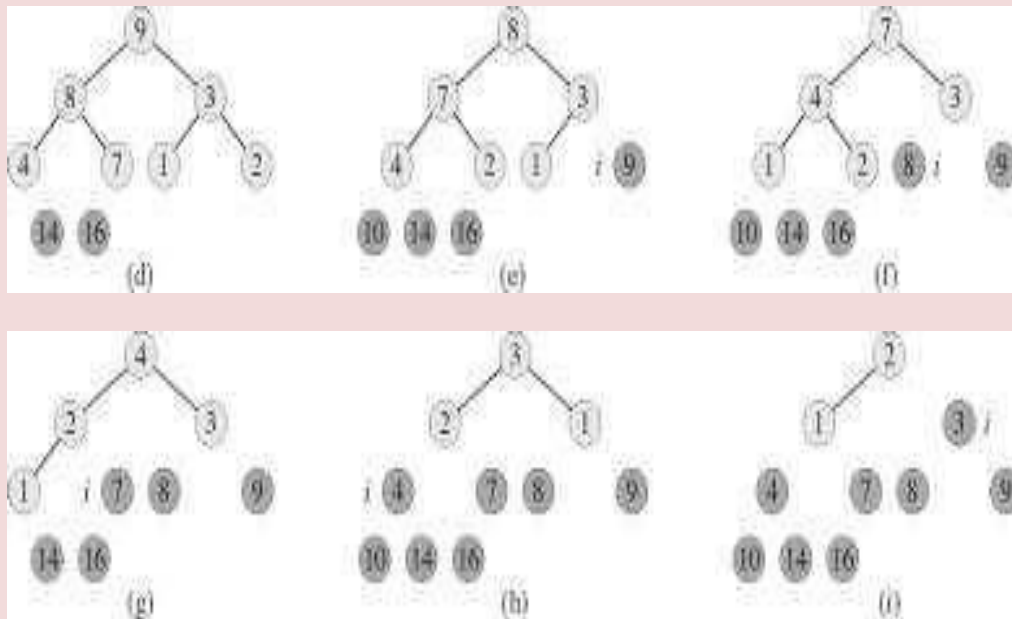
The total cost of BUILD-MAX-HEAP as being bounded is  $T(n)=O(n)$

### The HEAPSORT Algorithm

HEAPSORT(A)

1. BUILD MAX-HEAP(A)
2. for  $i=n$  to 2
3.     exchange  $A[1]$  with  $A[i]$
4.     MAX-HEAPIFY(A,1)





A 

1	2	3	4	7	8	9	10	14	16
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The HEAPSORT procedure takes time  $O(n \log n)$ , since the call to BUILD-MAX-HEAP takes time  $O(n)$  and each of the  $n - 1$  calls to MAX-HEAPIFY takes time  $O(\log n)$ .

### Lower Bounds For Sorting

**Review of Sorting:** So far we have seen a number of algorithms for sorting a list of numbers in ascending order. Recall that an *in-place* sorting algorithm is one that uses no additional array storage (however, we allow Quick sort to be called in-place even though they need a stack of size  $O(\log n)$  for keeping track of the recursion). A sorting algorithm is *stable* if duplicate elements remain in the same relative position after sorting.

**Slow Algorithms:** Include Bubble Sort, Insertion Sort, and Selection Sort. These are all simple  $\Theta(n^2)$ -in-place sorting algorithms. Bubble Sort and Insertion Sort can be implemented as stable algorithms, but Selection Sort cannot (without significant modifications).

**Merge sort:** Merge sort is a stable  $\Theta(n \log n)$  sorting algorithm. The downside is that Merge Sort is the only algorithm of the three that requires additional array storage, implying that it is in-place algorithm.

**Quick sort:** Widely regarded as the *fastest* of the fast algorithms. This algorithm is  $O(n \log n)$  in the *expected case*, and  $O(n^2)$  in the worst case. The probability that the algorithm takes asymptotically longer (assuming that the pivot is chosen randomly) is extremely small for large  $n$ . It is an (almost) in-place sorting algorithm but is not stable.

**Heap sort:** Heap sort is based on a nice data structure, called a *heap*, which is a fast priority queue. Elements can be inserted into a heap in  $O(\log n)$  time, and the largest item can be extracted in  $O(\log n)$  time. (It is also easy to set up a heap for extracting the smallest item.) If you only want to extract the  $k$  largest values, a heap can allow you to do this in  $O(n + k \log n)$  time. It is an in-place algorithm, but it is not stable.

**Lower Bounds for Comparison-Based Sorting:** Can we sort faster than  $O(n \log n)$  time?

We will give an argument that if the sorting algorithm is based solely on making comparisons between the keys in the array, then it is impossible to sort more efficiently than  $(n \log n)$  time. Such an algorithm is called a *comparison-based* sorting algorithm, and includes all of the algorithms given above. Virtually all known general purpose sorting algorithms are based on making comparisons, so this is not a very restrictive assumption. This does not preclude the possibility of a sorting algorithm whose sections are determined by other types of operations, for example, consulting the individual bits of numbers, performing arithmetic operations, indexing into an array based on arithmetic operations on keys. We will show that any *comparison-based* sorting algorithm for an input sequence  $a_1; a_2; \dots; a_n$  must make at least  $(n \log n)$  comparisons in the worst-case. This is still a difficult task if you think about it. It is easy to show that a problem *can* be solved fast (just give an algorithm). But to show that a problem *cannot* be solved fast you need to reason in some way about all the possible algorithms that might ever be written. In fact, it seems surprising that you could even hope to prove such a thing. The catch here is that we are limited to using comparison-based algorithms, and there is a clean mathematical way of characterizing all such algorithms.

**Decision Tree Argument:** In order to prove lower bounds, we need an abstract way of modeling “any possible” comparison-based sorting algorithm, we model such algorithms in terms of an abstract model called a *decision tree*. In a *comparison-based* sorting algorithm only comparisons between the keys are used to determine the action of the algorithm. Let  $a_1; a_2; \dots; a_n$  be the input sequence. Given two elements,  $a_i$  and  $a_j$ , their relative order can only be determined by the results of comparisons like  $a_i < a_j$ ,  $a_i \leq a_j$ ,  $a_i = a_j$ ,  $a_i \geq a_j$ , and  $a_i > a_j$ . A decision tree is a mathematical representation of a sorting algorithm (for a fixed value of  $n$ ). Each node of the decision tree represents a comparison made in the algorithm (e.g.,  $a_4 < a_7$ ), and the two branches represent the possible results, for example, the left sub tree consists of the remaining comparisons made under the assumption that  $a_4 \leq a_7$  and the right sub tree for  $a_4 > a_7$ . (Alternatively, one might be labeled with  $a_4 < a_7$  and the other with  $a_4 \geq a_7$ .) Observe that once we know the value of  $n$ , then the “action” of the sorting algorithm is completely determined by the results of its comparisons. This action may involve moving elements around in the array, copying them to other locations in memory, performing various arithmetic operations on non-key data. But the bottom-line is that at the end of the algorithm, the keys will be permuted in the final array in some way. Each leaf in the decision tree is labeled with the final permutation that the algorithm generates after making all of its comparisons. To make this more concrete, let us assume that  $n = 3$ , and let’s build a decision tree for Selection Sort. Recall that the algorithm consists of two phases. The first finds the smallest element of the entire list, and swaps it with the first element. The second finds the smaller of the remaining two items, and swap it with the second element. Here is the decision tree (in outline form). The first comparison is between  $a_1$  and  $a_2$ . The possible results are:

$a_1 \leq a_2$ : Then  $a_1$  is the current minimum. Next we compare  $a_1$  with  $a_3$  whose results might be either:

$a_1 \leq a_3$ : Then we know that  $a_1$  is the minimum overall, and the elements remain in their original positions. Then we pass to phase 2 and compare  $a_2$  with  $a_3$ . The possible results are:

$a_2 \leq a_3$ : Final output is  $a_1; a_2; a_3$ .

$a_2 > a_3$ : These two are swapped and the final output is  $a_1; a_3; a_2$ .

$a_1 > a_3$ : Then we know that  $a_3$  is the minimum is the overall minimum, and it is swapped with  $a_1$ . Then we pass to phase 2 and compare  $a_2$  with  $a_1$  (which is now in the third position of the array) yielding either:

$a_2 \leq a_1$ : Final output is  $a_3; a_2; a_1$ .

$a_2 > a_1$ : These two are swapped and the final output is  $a_3; a_1; a_2$ .

$a_1 > a_2$ : Then  $a_2$  is the current minimum. Next we compare  $a_2$  with  $a_3$  whose results might be either:

$a_2 \leq a_3$ : Then we know that  $a_2$  is the minimum overall. We swap  $a_2$  with  $a_1$ , and then pass to phase 2, and compare the remaining items  $a_1$  and  $a_3$ . The possible results are:

$a_1 \leq a_3$ : Final output is  $a_2; a_1; a_3$ .

$a_1 > a_3$ : These two are swapped and the final output is  $a_2; a_3; a_1$ .

$a_2 > a_3$ : Then we know that  $a_3$  is the minimum is the overall minimum, and it is swapped with  $a_1$ . We pass to phase 2 and compare  $a_2$  with  $a_1$  (which is now in the third position of the array) yielding either:

$a_2 \leq a_1$ : Final output is  $a_3; a_2; a_1$ .

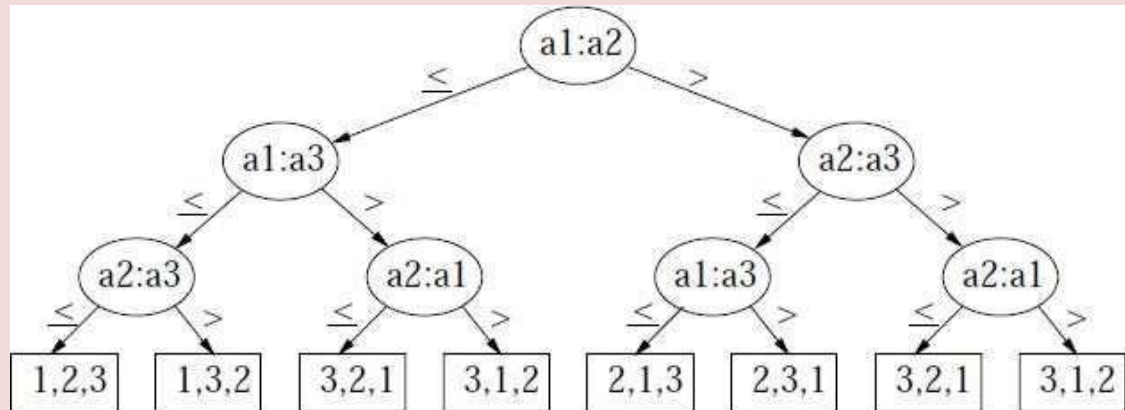
$a_2 > a_1$ : These two are swapped and the final output is  $a_3; a_1; a_2$ .

The final decision tree is shown below. Note that there are some nodes that are unreachable. For example, in order to reach the fourth leaf from the left it must be that  $a_1 \leq a_2$  and  $a_1 > a_2$ , which cannot both be true. Can you explain this? (The answer is that virtually all sorting algorithms, especially inefficient ones like selection sort, may make comparisons that are redundant, in the sense that their outcome has already been determined by earlier comparisons.) As you can see, converting a complex sorting algorithm like Heap Sort into a decision tree for a large value of  $n$  will be very tedious and complex, but I hope you are convinced by this exercise that it can be done in a simple mechanical way.



**(Decision Tree for Selection Sort on 3 keys.)**

**Using Decision Trees for Analyzing Sorting:** Consider any sorting algorithm. Let  $T(n)$  be the maximum number of comparisons that this algorithm makes on any input of size  $n$ . Notice that the running time for the algorithm must be at least as large as  $T(n)$ , since we are not counting data movement or other



computations at all. The algorithm defines a decision tree. Observe that the height of the decision tree is exactly equal to  $T(n)$ , because any path from the root to a leaf corresponds to a sequence of comparisons made by the algorithm.

As we have seen earlier, any binary tree of height  $T(n)$  has at most  $2^{T(n)}$  leaves. This means that this sorting algorithm can *distinguish* between at most  $2^{T(n)}$  different final actions. Let's call this quantity  $A(n)$ , for the number of different final actions the algorithm can take. Each action can be thought of as a specific way of permuting the original input to get the sorted output. How many possible actions must any sorting algorithm distinguish between? If the input consists of  $n$  distinct numbers, then those numbers could be presented in any of  $n!$  Different permutations. For each different permutation, the algorithm must "unscramble" the numbers in an essentially different way, that is it must take a different action, implying that  $A(n) \geq n!$ . (Again,  $A(n)$  is usually not exactly equal to  $n!$  because most algorithms contain some redundant unreachable leaves.)

Since  $A(n) \leq 2^{T(n)}$  we have  $2^{T(n)} \geq n!$ , implying that

$$T(n) \geq \lg(n!):$$

We can use *Stirling's approximation* for  $n!$  yielding:

$$n! \geq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

$$T(n) \geq \log\left(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n\right)$$

$$= \log\sqrt{2\pi n} + n\log n - n\log e \in \Omega(n\log n)$$

Thus we have the following theorem.

**Theorem:** Any comparison-based sorting algorithm has worst-case running time  $(n \log n)$ .

This can be generalized to show that the *average-case* time to sort is also  $(n \log n)$  (by arguing about the average height of a leaf in a tree with at least  $n!$  leaves). The lower bound on sorting can be generalized to provide lower bounds to a number of other problems as well.

## MODULE -II

### GREEDY METHOD

#### Introduction

Let we are given a problem to sort the array  $a = \{5, 3, 2, 9\}$ . Someone says the array after sorting is  $\{1, 3, 5, 7\}$ . Can we consider the answer is correct? The answer is definitely “no” because the elements of the output set are not taken from the input set. Let someone says the array after sorting is  $\{2, 5, 3, 9\}$ . Can we admit the answer? The answer is again “no” because the output is not satisfying the objective function that is the first element must be less than the second, the second element must be less than the third and so on. Therefore, the solution is said to be a feasible solution if it satisfies the following constraints.

- (i) **Explicit constraints:** - The elements of the output set must be taken from the input set.
- (ii) **Implicit constraints:**-The objective function defined in the problem.

The best of all possible solutions is called the optimal solution. In other words we need to find the solution which has the optimal (maximum or minimum) value satisfying the given constraints.

The Greedy approach constructs the solution through a sequence of steps. Each step is chosen such that it is the best alternative among all feasible choices that are available. The choice of a step once made cannot be changed in subsequent steps.

Let us consider the problem of coin change. Suppose a greedy person has some 25p, 20p, 10p, 5paise coins. When someone asks him for some change then he wants to give the change with minimum number of coins. Now, let someone requests for a change of 45p then he first selects 25p. Then the remaining amount is 20p. Next, he selects the largest coin that is less than or equal to 20p i.e. 20p. The remaining 0p is paid by selecting a 0p coin. So the demand for 45p is paid by giving total 2 numbers of coins. This solution is an optimal solution. Now, let someone requests for a change of 40p then the Greedy approach first selects 25p coin, then a 10p coin and finally a 5p coin. However, the same could be paid with two 20p coins. So it is clear from this example that Greedy approach tries to find the optimal solution by selecting the elements one by one that are locally optimal. But Greedy method never gives the guarantee to find the optimal solution.

The choice of each step in a greedy approach is done based on the following:

- It must be feasible
- It must be locally optimal
- It must be unalterable

#### Fractional Knapsack Problem

Let there are  $n$  number of objects and each object is having a weight and contribution to

profit. The knapsack of capacity  $M$  is given. The objective is to fill the knapsack in such a way that profit shall be maximum. We allow a fraction of item to be added to the knapsack.

Mathematically, we can write

$$\text{maximize } \sum_{i=1}^n p_i x_i$$

*Subject*  
*to*

$$\sum_{i=1}^n w_i x_i \leq M$$

$0 \leq x_i \leq 1$  for  $i = 1, 2, \dots, n$

Where  $p_i$  and  $w_i$  are the profit and weight of  $i^{\text{th}}$  object and  $x_i$  is the fraction of  $i^{\text{th}}$  object to be selected.

For example

$$\text{Given } n = 3, (p_1, p_2, p_3) = \{25, 24, 15\}$$

$$(w_1, w_2, w_3) = \{18, 15, 10\} \quad M = 20$$

\_\_\_\_\_ Solution

Some of the feasible solutions are shown in the following table.

<i>Solution No</i>	$x_1$	$x_2$	$x_3$	$\sum w_i x_i$	$\sum p_i x_i$
1	1	2/15	0	20	28.2
2	0	2/3	1	20	31.0
3	0	1	1/2	20	31.5

These solutions are obtained by different greedy strategies.



```

    cu = cu - w[i] ;
}
}
if( i <= n){
    x[i] = cu/w[i] ;
return x;
}

```

**Algorithm 1.** Greedy algorithm for fractional knapsack problem.

### Huffman Coding

Each character is represented in 8 bits when characters are coded using standard codes such as ASCII. It can be seen that the characters coded using standard codes have fixed-length code word Representation. In this fixed-length coding system the total code length is more. For example, let we have six characters (a, b, c, d, e, f) and their frequency of occurrence in a message is {45, 13, 12, 16, 9, 5}. In fixed-length coding system we can use three characters to represent each code. Then the total code length of the message is  $(45+13+12+16+9+5) \times 3 = 100 \times 3 = 300$ .

Let us encode the characters with variable-length coding system. In this coding system, the character with higher frequency of occurrence is assigned fewer bits for representation while the characters having lower frequency of occurrence in assigned more bits for representation. The variable length code for the characters are shown in the following table The total code length in variable length coding system is  $1 \times 45 + 3 \times 12 + 3 \times 16 + 4 \times 9 + 4 \times 5 = 224$ . Hence fixed length code requires 300 bits while variable code requires only 224 bits.

a	b	c	d	e	f
0	101	10	111	110	110
		0		1	0

### Prefix (Free) Codes

We have seen that using variable-length code word we minimize the overall encoded string length. But the question arises whether we can decode the string. If *a* is encoded 1 instead of 0 then the encoded string “111” can be decoded as “d” or “aaa”. It can be seen that we get ambiguous string. The key point to remove this ambiguity is to use prefix codes. Prefix codes is the code in which there is no codeword that is a prefix of other codeword.

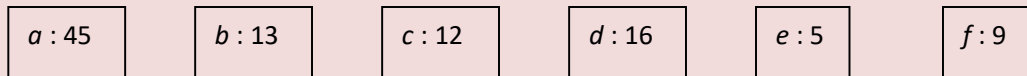
The representation of “decoding process” is binary tree whose leaves are characters. We interpret the binary codeword for a character as path from the root to that character, where

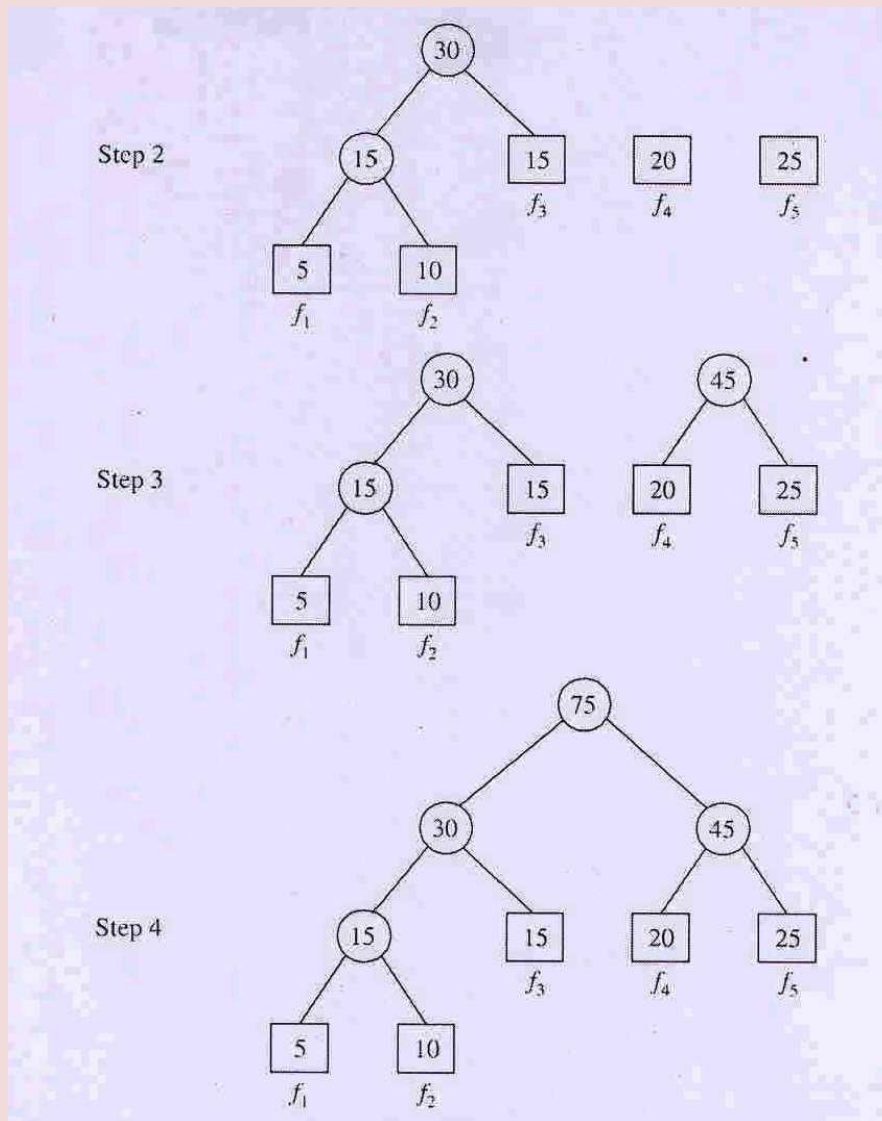
- “0” means “go to the left child”
- “1” means “go to the right child”

#### Greedy Algorithm for Huffman Code:

According to Huffman algorithm, a bottom up tree is built starting from the leaves. Initially, there are  $n$  singleton trees in the forest, as each tree is a leaf. The greedy strategy first finds two trees having minimum frequency of occurrences. Then these two trees are merged in a single tree where the frequency of this tree is the total sum of two merged trees. The whole process is repeated until there is only one tree in the forest.

Let us consider a set of characters  $S = \langle a, b, c, d, e, f \rangle$  with the following frequency of occurrences  $P = \langle 45, 13, 12, 16, 5, 9 \rangle$ . Initially, these six characters with their frequencies are considered six singleton trees in the forest. The step wise merging these trees to a single tree is shown in Fig. 6.3. The merging is done by selecting two trees with minimum frequencies till there is only one tree in the forest.





Step wise merging of the singleton trees.

Now the left branch is assigned a code “0” and right branch is assigned a code “1”. The decode tree after assigning the codes to the branches.

The binary codeword for a character is interpreted as path from the root to that character; Hence, the codes for the characters are as follows

$a = 0$

$b = 101$

$c = 100$

$d = 111$



$e = 1100$

$f = 1101$

Therefore, it is seen that no code is the prefix of other code. Suppose we have a code 01111001101. To decode the binary codeword for a character, we traverse the tree. The first character is 0 and the character at which the tree traversal terminates is  $a$ . Then, the next bit is 1 for which the tree is traversed right. Since it has not reached at the leaf node, the tree is next traversed right for the next bit

1. Similarly, the tree is traversed for all the bits of the code string. When the tree traversal terminates at a leaf node, the tree traversal again starts from the root for the next bit of the code string. The character string after decoding is “ $adcf$ ”.

---

---

**Algorithm** HUFFMAN( $n, S$ )

```
{
  //  $n$  is the number of symbols and  $S$  in the set of characters, for each character  $c \in S$ , the
  frequency of
  occurrence in  $f(c)$  //

  Initialize the priority queue;

   $Q = S$  ; // Initialize the priority  $Q$  with the frequencies of all the characters of set  $S$ //

  for( $i = 1$  ;  $i \leq n-i$ ,  $i++$ ){

     $z = \text{CREAT\_NODE}()$  ; // create a node pointed by  $z$ ; //

    // Delete the character with minimum frequency from the  $Q$  and store in
    node  $x$ // $x = \text{DELETE\_MIN}(Q)$ ;

    // Delete the character with next minimum frequency from the  $Q$  and store in
    node  $y$ // $y = \text{DELETE\_MIN}(Q)$ ;

     $z\text{left} = x$ ;           // Place  $x$  as the left child of  $z$ //
     $z\text{right} = y$ ;         // Place  $y$  as the right child of  $z$ //

    //The value of node  $z$  is the sum of values at node  $x$  and node  $y$ //

     $f(z) = f(x) + f(y)$ ;

    //insert  $z$  into the priority  $Q$ //

    INSERT ( $Q, z$ );

  }return DELETE_MIN( $Q$ )
}
```

**Algorithm 2.** Algorithm of Huffman coding.

Activity Selection Problem

Suppose we have a set of activities  $S = \{a_1, a_2, \dots, a_n\}$  that with to use a common resource. The objective is to schedule the activities in such a way that maximum number of activities can be performed. Each activity  $a_i$  has a start times  $s_i$  and a finish time  $f_i$ , where  $0 \leq s_i < f_i < \infty$ . The activities  $a_i$  and  $a_j$  are said to compatible if the intervals  $[s_i, f_i]$  and  $[s_j, f_j]$  do not overlap that means  $s_i \geq f_j$  or  $s_j \geq f_i$ .

For example, let us consider the following set  $S$  of activities, which are sorted in monotonically increasing order of finish time.

$i$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$	$a_9$	$a_{10}$	$a_{11}$			
			$s_i$	1	3	0	5	3	5	6	8	8	2	12
				$f_i$	4	5	6	7	8	9	10	11	7	2

For this example, the subsets  $\{a_3, a_9, a_{11}\}$ ,  $\{a_1, a_4, a_8, a_{11}\}$  and  $\{a_2, a_4, a_9, a_{11}\}$  consist of mutually compatible activities. We have two largest subsets of mutually compatible activities.

Now, we can devise greedy algorithm that works in a top down fashion. We assume that the  $n$  input activities are ordered by monotonically increasing finish time or it can be sorted into this order in  $O(n \log_2 n)$  time. The greedy algorithm for activity selection problem is given below.

---



---

**Algorithm**ACTIVITY SELECTION ( $S, f$ )

```

{
    n = LENGTH (S) ; // n is the total number of activities //
    A = {a1} ; // A is the set of selected activities and
    initialized to a1// i = 1; // i represents the recently selected
    activity //
    for (j = 2 ; j <= n ; j++)
    {
        if (sj ≥ fi) {
            A = A ∪ {am} ;
        }
    }
}

```

```
i=j;  
}
```

}

**Return A ;**

}

---

---

**Algorithm 3.** Algorithm of activity selection problem.

The algorithm takes the start and finish times of the activities, represented as arrays  $s$  and  $f$ , length

( $s$ ) gives the total number of activities. The set  $A$  stores the selected activities. Since the activities are ordered with respect to their finish times the set  $A$  is initialized to contain just the first activity  $a_1$ . The variable  $i$  stores the index of the recently selected activity. The for loop considers each activity  $a_j$  and adds to the set  $A$  if it is mutually compatible with the previously selected activities. To see whether activity  $a_j$  is compatible with every activity  $a_i$  in  $A$ , it needs to check whether the start time of  $a_j$  is greater or equal to the finish time of the recently selected activity  $a_i$ .

### Minimum Cost Spanning Tree

Let  $G = (V, E)$  be the graph where  $V$  is the set of vertices,  $E$  is the set of edges and  $|V| = n$ . The spanning tree  $G = (V, E)$  is a sub graph of  $G$  in which all the vertices of graph  $G$  are connected with minimum number of edges. The minimum number of edges required to connect all the vertices of a graph  $G$  is  $n - 1$ . Spanning tree plays a very important role in designing efficient algorithms.

Let us consider a graph shown in Fig 6.6(a). There are a number of possible spanning trees that is shown in Fig 6.6(b).

If we consider a weighted graph then all the spanning trees generated from the graph have different weights. The weight of the spanning tree is the sum of its edges weights. The spanning tree with minimum weight is called minimum spanning tree (MST). Fig. 6.7 shows a weighted graph and the minimum spanning tree.

A greedy method to obtain the minimum spanning tree would construct the tree edge by edge, where each edge is chosen according to some optimization criterion. An obvious criterion would be to choose an edge which adds a minimum weight to the total weight of the edges selected so far. There are two ways in which this criterion can be achieved.

1. The set of edges selected so far always forms a tree, the next edge to be added is such that not only it adds a minimum weight, but also forms a tree with the previous edges; it can be shown that the algorithm results in a minimum cost tree; this algorithm is called Prim's algorithm.
2. The edges are considered in non decreasing order of weight; the set  $T$  of edges at each stage is such that it is possible to complete  $T$  into a tree; thus  $T$  may not be a tree at all

stages of the algorithm; this also results in a minimum cost tree; this algorithm is called Kruskal's algorithm.

### Prim's Algorithm

This algorithm starts with a tree that has only one edge, the minimum weight edge. The edges  $(j, q)$  is added one by one such that node  $j$  is already included, node  $q$  is not included and weight  $wt(j, q)$  is the minimum amongst all the edges  $(x, y)$  for which  $x$  is in the tree and  $y$  is not. In order to execute this algorithm efficiently, we have a node index  $near(j)$  associated with each node  $j$  that is not yet included in the tree. If a node is included in the tree,  $near(j) = 0$ . The node  $near(j)$  is selected into the tree such that  $wt(j, near(j))$  is the minimum amongst all possible choices for  $near(j)$ .

### Algorithm PRIM ( $E, wt, n, T$ )

// $E$  is the set of edges,  $wt(n, n)$  is the weight adjacency matrix for  $G$ ,  $n$  is the number of nodes and  $T(n-1, 2)$  stores the spanning tree.

```
{
     $(k, l)$  = edge with minimum wt.
     $minwt = wt[k, l]$  ;
     $T[1, 1] = k, T[1, 2] = l$  ;
    for( $i = 1; i \leq n; i++$ ){
        if( $wt[i, k] < wt[i, l]$  )
             $near[i] = k$ ;
        else
             $near[i] = l$ ;
    }
     $near[k] = near[l] = 0$ ;
    for( $i = 2; i \leq n-1 ; i++$ )
    {
        let  $j$  be an index such that  $near[j] \neq 0$  and  $wt[j, near[j]]$  is minimum.
         $T[i, 1] = j; T[i, 2] = near[j]$ ;
    }
}
```

```

    minwt = minwt + wt[j,
near[j]] ;near[j] = 0 ;
for(k = 1; k<= n ; k++){
    if (near[k] 0 and wt[k, near[k]] >wt[k, j])
        near[k] = j;
    }
}
if(minwt == ∞)
    print("No spanning tree");

return minwt;
}

```

---

**Algorithm 4.** Prim’s algorithm for finding MST.

The weighted undirected graph to illustrate Prim’s algorithm

Let us consider the weighted undirected graph shown in Fig.6.8 and the objective is to construct a minimum spanning tree. The step wise operation of Prim’s algorithm is described as follows.

Step 1 The minimum weight edge is (2, 3) with weight 5. Hence, the edge (2, 3) is added to the tree.

$near(2)$  and  $near(3)$  are set 0.

Step 2 Find near of all the nodes that are not yet selected into the tree and its cost.

$near(1) = 2$      $weight = 16$

$near(4) = 2$      $weight =$

$6$      $near(5) = -$      $weight =$

$\infty$      $near(6) = 2$      $weight =$

11

The node 4 is selected and the edge (2, 4) is added to the tree because

$weight(4, near(4))$  is minimum. Then  $near(4)$  is set 0.

Step 3       $near(1) = 2$      $weight = 16$

$near(5) = 4$      $weight = 18$

$near(6) = 2$      $weight = 11$

As  $weight(6, near(6))$  is minimum, the node 6 is selected and edge (2, 6) is added to the tree. So

$near(6)$  is set 0

Step 4       $near(1) = 2$      $weight = 16$

$near(5) = 4$      $weight = 18$

Next, the edge (2, 1) is added to the tree as  $weight(1, near(1))$  is minimum. So  $near(1)$  is set 0

Step 5       $near(5) = 1$      $weight = 12$

The edge (1, 5) is added to the tree. The Fig. 6.9(a) to 6.9(e) show the step wise construction of MST by Prim's algorithm.

Step wise construction of MST by Prim's algorithm

### **Time complexity of Prim's Algorithm**

Prim's algorithm has three for loops. The first *for* loop finds the near of all nodes which require  $O(n)$  time. The second *for* loop is to find the remaining  $n-2$  edges and the third *for* loop updates near of each node after adding a vertex to MST. Since the third for loop is within the second for loop, it requires  $O(n^2)$  time. Hence, the overall time complexity of Prim's algorithm is  $O(n^2)$ .

### **Kruskal's Algorithm**

This algorithm starts with a list of edges sorted in non decreasing order of weights. It repeatedly adds the smallest edge to the spanning tree that does not create a cycle. Initially, each vertex is in its own tree in the forest. Then, the algorithm considers each edge ordered by increasing weights. If the edge  $(u, v)$  connects two different trees, then  $(u, v)$  is added to

the set of edges of the MST and two trees connected by an edge  $(u, v)$  are merged in to a single tree. If an edge  $(u, v)$  connects two vertices in the same tree, then edge  $(u, v)$  is discarded.

The Krushal's algorithm for finding the MST is presented as follows. It starts with an empty set  $A$ , and selects at every stage the shortest edge that has not been chosen or rejected regardless of where this edge is situated in the graph. The pseudo code of Kruskal's algorithm is given in Algorithm .

The operations an disjoint sets used for Krushal's algorithm is as

follows:  $Make\_set(v)$  : create a new set whose only member is pointed to  $v$ .  $Find\_set(v)$  : returns a pointer to the set containing  $v$ .

$Union(u, v)$  : unites the dynamic sets that contain  $u$  and  $v$  into a new set that is union of thesetwo sets.

---



---

#### Algorithm KRUSKAL $(V, E, W)$

//  $V$  is the set of vertices,  $E$  is the set of edges and  $W$  is the adjacency matrix to store the weights of the links. //

{

$A = ;$

**for** (each vertex  $u$  in

$V$ )  $Make\_set(u)$

Create a min heap from the weights of the links using procedure *heapify*.

**for** (each least weight edge  $(u, v)$  in  $E$ ) // least weight edge is the root of

the heap // if  $(Find\_set(u) \neq Find\_set(v))$  { //  $u$  and  $v$  are in two different

sets //

$A = A \cup \{u, v\}$

$Union(u, v)$

}

}

return  $A$  ;



}

---

**Algorithm. 5** Kruskal's algorithm for finding MST.

Let us consider the graph shown in Fig.6.10 to illustrate Kruskal's algorithm.

The step wise procedure to construct MST by following the procedure presented given below.

Edge selected	Disjoint sets	Spanning tree after the edge included
—	{1} {2} {3} {4} {5} {6} {7}	$\Phi$
(1, 2)	{1, 2} {3} {4} {5} {6} {7}	
(2, 3)	{1, 2, 3} {4} {5} {6} {7}	
(4, 5)	{1, 2, 3} {4, 5} {6} {7}	
(6, 7)	{1, 2, 3} {4, 5} {6, 7}	
(1, 4)	{1, 2, 3, 4, 5} {6, 7}	
(2, 5)	Since 2 and 5 belong to the same set, the edge (2, 5) is discarded	
(4, 7)	{1, 2, 3, 4, 5, 6, 7}	
(3, 5)		
(2, 4)		
(3, 6)	Discarded	
(5, 6)	Discarded	

Step wise construction of MST by Kruskal's algorithm

### Time complexity of Kruskal's Algorithm

The Kruskal's algorithm first creates  $n$  trees from  $n$  vertices which is done in  $O(n)$  time. Then, a heap is created in  $O(n)$  time using heapify procedure. The least weight edge is at the root of the heap. Hence, the edges are deleted one by one from the heap and either added to the MST or discarded if it forms a cycle. This deletion process requires  $O(n \log_2 n)$ . Hence, the time complexity of Kruskal's algorithm is  $O(n \log_2 n)$ .

### Shortest Path Problem

Let us consider a number of cities connected with roads and a traveler wants to travel from his home city A to the destination B with a minimum cost. So the traveler will be interested to know the following:

- Is there a path from city A to city B?
- If there is more than one path from A to B, which is the shortest or least cost path?

Let us consider the graph  $G = (V, E)$ , a weighting function  $w(e)$  for the edges in  $E$  and a source node  $v_0$ . The problem is to determine the shortest path from  $v_0$  to all the remaining nodes of  $G$ . The solution to this problem is suggested by E.W. Dijkstra and the algorithm is popularly known as Dijkstra's algorithm.

This algorithm finds the shortest paths one by one. If we have already constructed  $i$  shortest paths, then the next path to be constructed should be the next shortest path. Let  $S$  be the set of vertices to which the shortest paths have already been generated. For  $z$  not in  $S$ , let  $dist[z]$  be the length of the shortest path starting from  $v_0$ , going through only those vertices that are in  $S$  and ending at  $z$ . Let  $u$  is the vertex in  $S$  to which the shortest path has already been found. If  $dist[z] > dist[u] + w(u,z)$  then  $dist[z]$  is updated to  $dist[u] + w(u,z)$  and the predecessor of  $z$  is set to  $u$ . The Dijkstra's algorithm is presented in Algorithm 6.6.

**Algorithm Dijkstra** ( $v_0, W, dist, n$ )

//  $v_0$  is the source vertex,  $W$  is the adjacency matrix to store the weights of the links,  $dist[k]$  is the array to store the shortest path to vertex  $k$ ,  $n$  is the number of vertices//

```
{
    for (i = 1 ; i <= n ; i++){
        S[i] = 0;           // Initialize the set S to empty i.e. i is not inserted into the set//
        dist[i] = w(v0,i) ; //Initialize the distance to each node
    }

    S[v0] = 1; dist[v0] = 0;
    for (j = 2; j<= n; j++) {
        choose a vertex u from those vertices not in S such that dist [u] is minimum.

        S[u] = 1;

        for (each z adjacent to u with S[z] = 0) {
            if(dist[z] >dist [u] + w[u, z])
                dist[z] = dist [u] + w[u, z];
        }
    }
}
```

}

**Algorithm :** Dijkstra's algorithm for finding shortest path.

Let us consider the graph. The objective is to find the shortest path from source vertex 0 to the allremaining nodes.

Iteration	Set of nodes to which the shortestpath is found	Vertex selected	Distance				
			0	1	2	3	4
Initial	{0}	-	0	10	$\infty$	5	$\infty$
1	{0, 3}	3	0	8	14	5	7
2	{0, 3,4}	4	0	8	13	5	7
3	{0, 3, 4, 1}	1	0	8	9	5	7
4	{0, 3, 4, 1, 2}	2	0	8	9	5	7

The time complexity of the algorithm is  $O(n^2)$ .

## Dynamic Programming

### Introduction

The Dynamic Programming (DP) is the most powerful design technique for solving optimization problems. It was invented by mathematician named Richard Bellman inn 1950s. The DP in closely related to divide and conquer techniques, where the problem is divided into smaller sub-problems and each sub-problem is solved recursively. The DP differs from divide and conquer in a way that instead of solving sub-problems recursively, it solves each of the sub-problems only once and stores the solution to the sub-problems in a table. The solution to the main problem is obtained by the solutions of these sub- problems.

The steps of Dynamic Programming technique are:

- **Dividing the problem into sub-problems:** The main problem is divided into smaller sub- problems. The solution of the main problem is expressed in terms of the solution for the smaller sub-problems.
- **Storing the sub solutions in a table:** The solution for each sub-problem is stored in a table sothat it can be referred many times whenever required.
- **Bottom-up computation:** The DP technique starts with the smallest problem instance and develops the solution to sub instances of longer size and finally obtains the solution of the original problem instance.

The strategy can be used when the process of obtaining a solution of a problem can be viewed as a sequence of decisions. The problems of this type can be solved by taking an optimal sequence of decisions. An optimal sequence of decisions is found by taking one decision at a time and never making an erroneous decision. In Dynamic Programming, an optimal sequence of decisions is arrived at by using the principle of optimality. *The principle of optimality states that whatever be the initial state and decision, the remaining decisions*

must constitute an optimal decision sequence with regard to the state resulting from the first decision.

A fundamental difference between the greedy strategy and dynamic programming is that in the greedy strategy only one decision sequence is generated, whereas in the dynamic programming, a number of them may be generated. Dynamic programming technique guarantees the optimal solution for a problem whereas greedy method never gives such guarantee.

### Matrix chain Multiplication

Let, we have three matrices  $A_1$ ,  $A_2$  and  $A_3$ , with order  $(10 \times 100)$ ,  $(100 \times 5)$  and  $(5 \times 50)$  respectively.

Then the three matrices can be multiplied in two ways.

- (i) First, multiplying  $A_2$  and  $A_3$ , then multiplying  $A_1$  with the resultant matrix i.e.  $A_1(A_2 A_3)$ .
- (ii) First, multiplying  $A_1$  and  $A_2$ , and then multiplying the resultant matrix with  $A_3$  i.e.  $(A_1 A_2) A_3$ .

The number of scalar multiplications required in case 1 is  $100 * 5 * 50 + 10 * 100 * 50 = 25000 + 50,000 = 75,000$  and the number of scalar multiplications required in case 2 is  $10 * 100 * 5 + 10 * 5 * 50 = 5000 + 2500 = 7500$

To find the best possible way to calculate the product, we could simply parenthesize the expression in every possible fashion and count each time how many scalar multiplications are required. Thus the matrix chain multiplication problem can be stated as “*find the optimal parenthesisation of a chain of matrices to be multiplied such that the number of scalar multiplications is minimized*”.

### Dynamic Programming Approach for Matrix Chain Multiplication

Let us consider a chain of  $n$  matrices  $A_1, A_2, \dots, A_n$ , where the matrix  $A_i$  has dimensions  $P[i-1] \times P[i]$ .

Let the parenthesisation at  $k$  results two sub chains  $A_1, \dots, A_k$  and  $A_{k+1}, \dots, A_n$ . These two sub chains must

each be optimal for  $A_1, \dots, A_n$  to be optimal. The cost of matrix chain  $(A_1, \dots, A_n)$  is calculated as

$cost(A_1, \dots, A_k) + cost(A_{k+1}, \dots, A_n) + \text{cost of multiplying two resultant matrices together i.e.}$

$cost(A_1, \dots, A_n) = cost(A_1, \dots, A_k) + cost(A_{k+1}, \dots, A_n) + \text{cost of multiplying two resultant matrices together.}$

Here, the cost represents the number of scalar multiplications. The sub chain  $(A_1, \dots, A_k)$  has a dimension  $P[0] \times P[k]$  and the sub chain  $(A_{k+1}, \dots, A_n)$  has a dimension  $P[k] \times P[n]$ . The

number of scalar multiplications required to multiply two resultant matrices is  $P[0] \times P[k] \times P[n]$ .

Let  $m[i, j]$  be the minimum number of scalar multiplications required to multiply the matrix chain  $(A_i \dots A_j)$ . Then

- (i)  $m[i, j] = 0$  if  $i = j$
- (ii)  $m[i, j] = \min_{i < k < j} (\text{minimum number of scalar multiplications required to multiply } (A_i \dots A_k) + \text{minimum number of scalar multiplications required to multiply } (A_{k+1} \dots A_j) + \text{cost of multiplying two resultant matrices i.e. } P[i-1] \times P[k] \times P[j])$

However, we don't know the value of  $k$ , for which  $m[i, j]$  is minimum. Therefore, we have to try all  $j - i$  possibilities.

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i < k < j} (m[i, k] + m[k, j] + P[i-1] \times P[k] \times P[j]) & \text{Otherwise} \end{cases}$$

Therefore, the minimum number of scalar multiplications required to multiply  $n$  matrices  $A_1, A_2, \dots, A_n$  is

$$m[1, n] = \min_{1 \leq k \leq n} m[1, k] + m[k, n] + P[0] P[k] P[n]$$

The dynamic programming approach for matrix chain multiplication is presented in Algorithm.

**Algorithm MATRIX-CHAIN-MULTIPLICATION (P)**

// P is an array of length  $n+1$  i.e. from  $P[0]$  to  $P[n]$ . It is assumed that the matrix  $A_i$  has the dimension  $P[i-1] \times P[i]$ .

```

{
    for(i = 1; i <= n;
        i++) m[i, i] =
        0;
    for(l = 2; l <= n; l++){
        for(i = 1; i <= n-(l-1);
            i++){ j = i + (l-1);
            m[i, j] = ∞;
            for(k = i; k <= j-1; k++)
                q = m[i, k] + m[k+1, j] + P[i-1] P[k] P[j] ;
                if (q < m [i, j]){
                    m[i, j] = q;
                    s[i, j] = k;
                }
            }
        }
    }
}

return m and s.

```



}  


---

**Algorithm 7.2** Matrix Chain multiplication algorithm.

Now let us discuss the procedure and pseudo code of the matrix chain multiplication. Suppose, we are given the number of matrices in the chain is  $n$  i.e.  $A_1, A_2, \dots, A_n$  and the dimension of matrix  $A_i$  is  $P[i-1] \times P[i]$ . The input to the matrix-chain-order algorithm is a sequence  $P[n+1] = \{P[0], P[1], \dots, P[n]\}$ . The algorithm first computes  $m[i, i] = 0$  for  $i = 1, 2, \dots, n$  in lines 2-3. Then, the algorithm computes  $m[i, j]$  for  $j - i = 1$  in the first step to the calculation of  $m[i, j]$  for  $j - i = n - 1$  in the last step. In lines 3 - 11, the value of  $m[i, j]$  is calculated for  $j - i = 1$  to  $j - i = n - 1$  recursively. At each step of the calculation of  $m[i, j]$ , a calculation on  $m[i, k]$  and  $m[k+1, j]$  for  $ik < j$ , are required, which are already calculated in the previous steps.

To find the optimal placement of parenthesis for matrix chain multiplication  $A_i, A_{i+1}, \dots, A_j$ , we should test the value of  $k < j$  for which  $m[i, j]$  is minimum. Then the matrix chain can be divided from  $(A_1 \dots A_k)$  and  $(A_{k+1} \dots A_j)$ .

Let us consider matrices  $A_1, A_2, \dots, A_5$  to illustrate MATRIX-CHAIN-MULTIPLICATION algorithm. The matrix chain order  $P = \{P_0, P_1, P_2, P_3, P_4, P_5\} = \{5, 10, 3, 12, 5, 50\}$ . The objective is to find the minimum number of scalar multiplications required to multiply the 5 matrices and also find the optimal sequence of multiplications.

The solution can be obtained by using a bottom up approach that means first we should calculate  $m_{ii}$  for  $1 \leq i \leq 5$ . Then  $m_{ij}$  is calculated for  $j - i = 1$  to  $j - i = 4$ . We can fill the table shown to find the solution.

Table to store the partial solutions of the matrix chain multiplication problem

The value of  $m_{ii}$  for  $1 \leq i \leq 5$  can be filled as 0 that means the elements in the first row can be assigned 0. Then

For  $j - i = 1$

$$m_{12} = P_0 P_1 P_2 = 5 \times 10 \times 3 = 150$$

$$m_{23} = P_1 P_2 P_3 = 10 \times 3 \times 12 = 360$$

$$m_{34} = P_2 P_3 P_4 = 3 \times 12 \times 5 = 180$$

$$m_{45} = P_3 P_4 P_5 = 12 \times 5 \times 50 = 3000$$

For  $j - i = 2$

$$m_{13} = \min \{m_{11} + m_{23} + P_0 P_1 P_3, m_{12} + m_{33} + P_0 P_2 P_3\}$$

$$= \min \{0 + 360 + 5 * 10 * 12, 150 + 0 + 5*3*12\}$$

$$= \min \{360 + 600, 150 + 180\} = \min \{960,$$

$$330\} = 330$$

$$m_{24} = \min \{m_{22} + m_{34} + P_1 P_2 P_4, m_{23}$$

$$\begin{aligned}
& + m_{44} + P_1 P_3 P_4 \} \\
& = \min \{ 0 + 180 + 10 \cdot 3 \cdot 5, 360 + 0 + 10 \cdot 12 \cdot 5 \} \\
& = \min \{ 180 + 150, 360 + 600 \} = \min \{ 330, \\
960 \} = 330 & m_{35} = \min \{ m_{33} + m_{45} + P_2 P_3 P_5, m_{34} \\
& + m_{55} + P_2 P_4 P_5 \} \\
& = \min \{ 0 + 3000 + 3 \cdot 12 \cdot 50, 180 + 0 + 3 \cdot 5 \cdot 50 \} \\
& = \min \{ 3000 + 1800 + 180 + 750 \} = \min \{ 4800, 930 \} = 930
\end{aligned}$$

For  $j - i = 3$

$$\begin{aligned}
m_{14} & = \min \{ m_{11} + m_{24} + P_0 P_1 P_4, m_{12} + m_{34} + P_0 P_2 P_4, m_{13} + m_{44} + P_0 P_3 P_4 \} \\
& = \min \{ 0 + 330 + 5 \cdot 10 \cdot 5, 150 + 180 + 5 \cdot 3 \cdot 5, 330 + 0 + 5 \cdot 12 \cdot 5 \} \\
& = \min \{ 330 + 250, 150 + 180 + 75, 330 + 300 \} \\
& = \min \{ 580, 405, 630 \} = 405
\end{aligned}$$

$$\begin{aligned}
m_{25} & = \min \{ m_{22} + m_{35} + P_1 P_2 P_5, m_{23} + m_{45} + P_1 P_3 P_5, m_{24} + m_{55} + P_1 P_4 P_5 \} \\
& = \min \{ 0 + 930 + 10 \cdot 3 \cdot 50, 360 + 3000 + 10 \cdot 12 \cdot 50, 330 + 0 + 10 \cdot 5 \cdot 50 \} \\
& = \min \{ 930 + 1500, 360 + 3000 + 6000, 330 + 2500 \} \\
& = \min \{ 2430, 9360, 2830 \} = 2430
\end{aligned}$$

For  $j - i = 4$

$$\begin{aligned}
m_{15} & = \min \{ m_{11} + m_{25} + P_0 P_1 P_5, m_{12} + m_{35} + P_0 P_2 P_5, m_{13} + m_{45} + P_0 P_3 P_5, m_{14} + m_{55} + P_0 \\
& P_4 P_5 \} \\
& = \min \{ 0 + 2430 + 5 \cdot 10 \cdot 50, 150 + 930 + 5 \cdot 3 \cdot 50, \\
& 330 + 3000 + 5 \cdot 12 \cdot 50, 405 + 0 + 5 \cdot 5 \cdot 50 \} \\
& = \min \{ 2430 + 2500, 150 + 930 + 750, 330 + 3000 + 3000, 405 + 1250 \} \\
& = \min \{ 4930, 1830, 6330, 1655 \} = 1655
\end{aligned}$$

Hence, minimum number of scalar multiplications required to multiply the given five matrices is 1655.

To find the optimal parenthesization of  $A_1 \dots A_5$ , we find the value of  $k$  is 4 for which  $m_{15}$  is minimum. So the matrices can be splitted to  $(A_1 \dots A_4) (A_5)$ . Similarly,  $(A_1 \dots A_4)$  can be splitted to  $(A_1 A_2) (A_3 A_4)$  because for  $k = 2$ ,  $m_{14}$  is minimum. No further splitting is required as the subchains  $(A_1 A_2)$  and

### Dynamic programming for LCS

**problem Theorem.4.1.(Optimal Structure of an LCS)**

Let  $X = x_1, x_2, \dots, x_n$  and  $Y = y_1, y_2, \dots, y_n$  be sequences and let  $Z = z_1, z_2, \dots, z_n$  be any LCS of X and Y.

**Case 1.** If  $x_m = y_n$ , then  $z_k = x_m = y_n$  and  $Z_{k-1}$  is an LCS of  $X_{m-1}$  and  $Y_{n-1}$ . **Case 2.** If  $x_m < y_n$ , then  $z_k = x_m$  implies that Z is an LCS

of  $X_{m-1}$  and Y. **Case 3.** If  $x_m > y_n$ , then  $z_k = y_n$  implies that Z is an LCS of X and  $Y_{n-1}$ .

Proof The proof of the theorem is presented below for all three cases.

**Case 1.** If  $x_m = y_n$  and we assume that  $z_k = x_m$  or  $z_k = y_n$  then  $x_m = y_n$  can be added to Z at any index after k violating the assumption that  $Z_k$  is the longest common subsequence. Hence  $z_k = x_m = y_n$ . If we do not consider  $Z_{k-1}$  as LCS of  $X_{m-1}$  and  $Y_{n-1}$ , then there may exist another subsequence W whose length is more than k-1. Hence, after adding  $x_m = y_n$  to the subsequence W increases the size of subsequence more than k, which again violates our assumption.

Hence,  $Z_k = x_m = y_n$  and  $Z_{k-1}$  must be an LCS of  $X_{m-1}$  and  $Y_{n-1}$ .

**Case 2.** If  $x_m < y_n$ , then  $z_k = x_m$  implies that Z is an LCS of  $X_{m-1}$  and  $Y_n$ . If there were a common subsequence W of  $X_{m-1}$  and Y with length greater than k then W would also be an LCS of  $X_m$  and  $Y_n$  violating our assumption that  $Z_k$  is an LCS of  $X_m$  and  $Y_n$ .

**Case 3.** The proof is symmetric to case-2.

Thus the LCS problem has an optimal structure.

**Overlapping Sub-problems**

From theorem, it is observed that either one or two cases are to be examined to find an LCS of  $X_m$  and  $Y_n$ . If  $x_m = y_n$ , then we must find an LCS of  $X_{m-1}$  and  $Y_{n-1}$ . If  $x_m < y_n$ , then we must find an LCS of  $X_{m-1}$  and  $Y_n$  and LCS of  $X_m$  and  $Y_{n-1}$ . The LCS of X and Y is the longer of these two LCSs. Let us define  $c[m, n]$  to be the length of an LCS of the sequences  $X_m$  and  $Y_n$ . The optimal structure of the LCS problem gives the recursive formula

$$c[m,n] = \begin{cases} 0 & \text{if } m = 0 \text{ or } n = 0 \\ c[m-1, n-1] + 1 & \text{if } x_m = y_n \\ \max\{c[m-1, n], c[m, n-1]\} & \text{if } x_m \neq y_n \end{cases} \quad \text{(I)}$$

Generalizing equation 7.1, we can formulate

$$c[i,j] = \begin{cases} 0 & \text{if } i=0 \text{ or } j=0 \\ \max\{c[i-1, j], c[i, j-1], c[i-1, j-1] + 1\} & \text{if } x_i = y_j \\ c[i, j-1] & \text{if } x_i \neq y_j \end{cases} \dots\dots\dots(\text{II})$$

### Computing the length of an LCS

Based on equation (I), we could write an exponential recursive algorithm but there are only  $m*n$  distinct problems. Hence, for the solution of  $m*n$  distinct subproblems, we use dynamic programming to compute the solution using bottom up approach.

The algorithm LCS\_length (X, Y) takes two sequences  $X = x_1, x_2, \dots, x_n$  and  $Y = y_1, y_2, \dots, y_n$  as inputs and find  $c[m, n]$  as the maximum length of the subsequence in X and Y. It stores  $c[i, j]$  and  $b[i, j]$  in tables  $c[m, n]$  and  $b[m, n]$  respectively, which simplifies the construction of optimal solution.

---



---

#### Algorithm LCS\_LENGTH (X, Y)

```

{
  m=length [X]
  n=length [Y]

  for( i =1; i<=m;
    i++)c[i,0] =
    0;
  for(j=0; j<n;
    j++)c[0, j]=
    0;
  for(i=1; i< m; i++){
    for(j = 1; j <= n; j++)
    {
      if(x[i] == y[j])
      {
        c[i, j] = 1 + c[i-1, j-1]; b[i, j] = '↖';
      }
    }
  }
}

```



}

```

else{
    if(c[i-1, j] >= c[i, j-1] )
        c[i, j] = c[i-1, j];
        b[i,j] = '□';
    else
        c[i,j] = c[i,
j-1];b[i, j] =
        '□'
    }
return c and b ;
}

```

---

**Algorithm :** Algorithm for finding Longest common subsequence .

### Constructing an LCS

The algorithm LCS\_LENGTH returns  $c$  and  $b$  tables. The  $b$  table can be used to construct the LCS of  $X$  and  $Y$  quickly.

---

**Algorithm** PRINT\_LCS ( $b, X, i, j$ )

```

{
    if (i == 0 || j == 0)
        return;
    if (b[i, j] == '↖') {
        PRINT_LCS (b, X, i-1, j-1)
        Print  $x_i$ 
    }
    else if (b[i, j] == '↓, ')

```

```

PRINT_LCS (b, X, i-1, j)

else
PRINT_LCS (b, X, i, j-1)

}

```

---

**Algorithm 7.4** Algorithm to print the Longest common subsequence .

Let us consider two sequences  $X = C, R, O, S$  and  $Y = R, O, A, D, S$  and the objective is to find the LCS and its length. The c and b table are computed by using the algorithm LCS\_LENGTH for X and Y that is shown. The longest common subsequence of X and Y is R, O, S and the length of LCS is 3.

### Reliability Design Problem

In this section, we present the dynamic programming approach to solve a problem with multiplicative constraints. Let us consider the example of a computer network in which a set of nodes are connected with each other. Let  $r_i$  be the reliability of a node i.e. the probability at

which the node forwards the packets correctly in  $r_i$ . Then the reliability of the path connecting from one node  $s$  to

another node  $d$  is  $r_i^k$  where  $k$  is the number of intermediate node. Similarly, we can also consider a

system with  $n$  devices connected in series, where the reliability of device  $i$  is  $r_i$ . The reliability of the system is  $r_i^k$ . For example if there are 5 devices connected in series and the reliability of each device

is 0.99 then the reliability of the system is  $0.99 \times 0.99 \times 0.99 \times 0.99 \times 0.99 = 0.951$ . Hence, it is desirable to connect multiple copies of the same devices in parallel through the use of switching circuits. The switching circuits determine the devices in any group functions properly. Then they make use of one such device at each stage.

Let  $m_i$  be the number of copies of device  $D_i$  in stage  $i$ . Then the probability that all  $m_i$  have malfunction i.e.  $(1-r_i)^{m_i}$ . Hence, the reliability of stage  $i$  becomes  $1-(1-r_i)^{m_i}$ . Thus, if  $r = 0.99$  and  $m = 2$ , the reliability of stage  $i$  is 0.9999. However, in practical situations it becomes less because the switching circuits are not fully reliable. Let us assume that the reliability of

stage  $i$  in  $\prod_{i=1}^k (1 - (1-r_i)^{m_i})$ ,  $i \leq n$ .

### Bellman Ford Algorithm

In the previous chapter, it is observed that the Dijkstra's algorithm finds the shortest path from one node to another node in a graph  $G = (V, E)$  where the weights of the links are

positive. However, if there are some negative weight edges then dijkstra's algorithm may fail to find the shortest path from one node to another. Hence, Bellman-ford algorithm solves the single source shortest paths problem in general case. Let us consider the graph  $G = (V, E)$  shown in Fig.7.10. Let us assume that node 1 is the source, node 2 and node 3 are destinations. Then by using Dijkstra's algorithm, we can compute the shortest path to node 2 and node 3 as 5 and 7 respectively whereas it is not actually the case. The shortest path from 1 to 3 is 123 and the path length is 2. This can be computed by Bellman-ford algorithm.

Before applying Bellman-ford algorithm, we assume that the negative weight edges are permitted but there should not be any negative weight cycle. This is necessary to answer that the shortest paths consists of a finite number of edges. The path from 1 to 3 is 121212...123 and the path length is  $-\infty$ . When there are no negative weight cycles, the shortest path between any two nodes in n-node graph contains  $n-1$  edges.

Let  $dist^l[u]$  be the length of the shortest path from source node  $v$  to the node  $u$  under the constraint that the shortest path contains at most  $l$  edges. The  $dist^l[u] = cost[v, u]$ , for  $1 \leq u \leq n$ . As we discussed earlier, if there is no negative weight cycle then the shortest path contains at most  $n-1$  edges. Hence,  $dist^{n-1}[u]$  is the length of the shortest path from  $v$  to  $u$ . Our goal is to compute  $dist^{n-1}[u]$  and this can be done by using Dynamic programming approach. We can make the following observations

- (i) If the shortest path from  $v$  to  $u$  with at most  $k$ ,  $k > 1$  edges has no more than  $k-1$  edges, then  $dist^k[u] = dist^{k-1}[u]$ .
- (ii) If the shortest path from  $v$  to  $u$  with at most  $k$ ,  $k > 1$  edges has exactly  $k$  edges, then it is made up of a shortest path from  $v$  to some vertex  $j$  followed by edge  $\langle j, u \rangle$ . The path from  $v$  to  $j$  has  $k-1$  edges and its length is  $dist^{k-1}[j]$ . All vertices  $i$  such that  $\langle i, u \rangle \in E$  are the candidates of  $j$ . Since we are interested in a shortest path, the  $i$  that minimizes  $dist^{k-1}[i] + cost[i, u]$  is the correct value for  $j$ .

$$Dist^k[u] = dist^{k-1}[u], \min_{(i, u) \in E} dist^{k-1}[i] + cost[i, u]$$



The Bellman ford algorithm is presented in Algorithm.

---

---

**Algorithm** BELLMAN FORD( $v, cost, dist, n$ )

// $v$  is the source,  $cost$  is the adjacency matrix representing the cost of edges,  $dist$  stores the distance to all nodes,  $n$  is the number of nodes//

```
{  
  
1  for( $i=1; i \leq n; i++$ )  
  
2   $dist[i]=cost[v][i];$   
  
3  for( $k=2; k \leq n-1; k++$ ){  
  
4      for each  $u$  such that  $k \neq v$  and  $(j, u) \in E$   
5          if( $dist[u] > dist[j] + cost[j][u]$ )  
  
6               $dist[u] = dist[j] + cost[j][u];$   
  
7  } //end for  $k$   
  
}
```

---

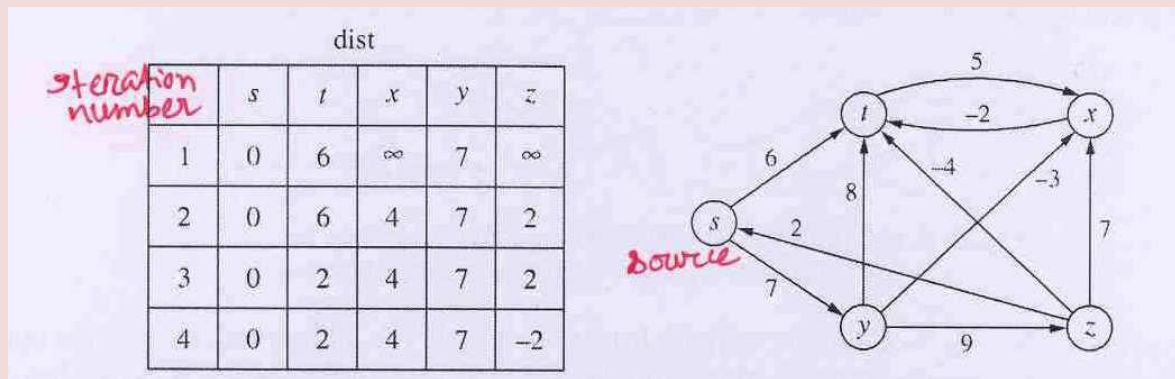
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**Algorithm :** Bellman Ford Algorithm.

The Bellman ford algorithm starts with computing the path from source to each node  $i$  in  $v$ . Since there are  $n$  nodes in a graph, the path to any node can contain at most  $n-1$  nodes. Then for each node  $u$ , remaining  $n-2$  nodes are required to be examined. If  $\langle j, u \rangle \in E$  then the condition  $dist[u]$  is compared

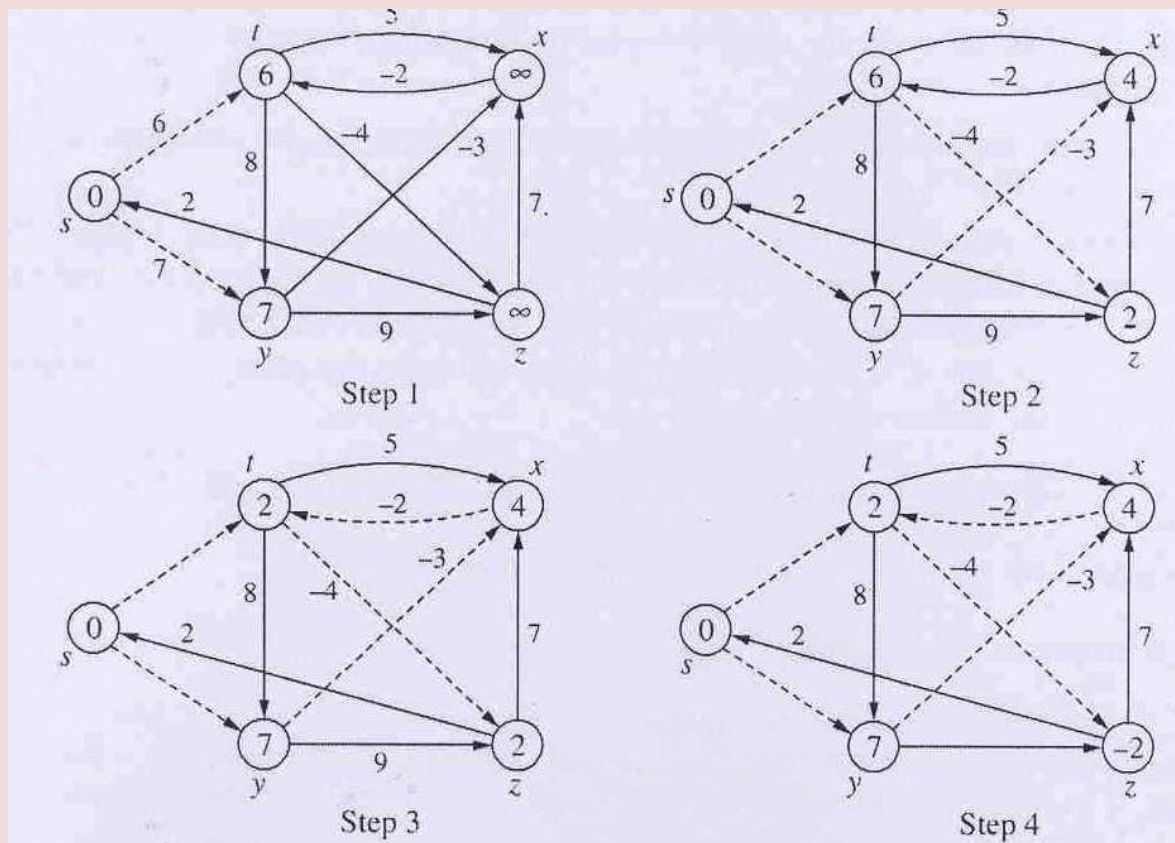
with  $dist[j] + cost[j][u]$ . If  $dist[u]$  is greater than  $dist[j] + cost[j][u]$  then  $dist[u]$  is updated to  $dist[j] + cost[j][u]$ .

Let us consider the example network shown in Fig. 7.12.



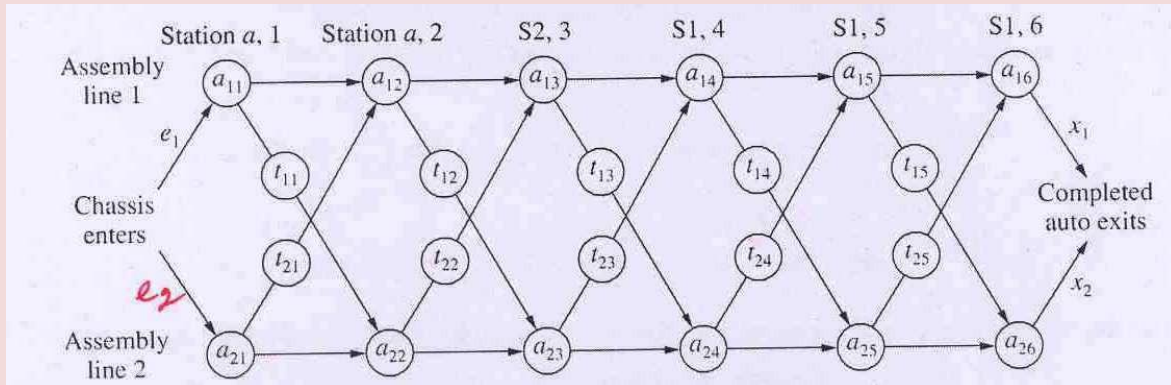
**The example graph and its adjacency matrix**

The time complexity of Bellman ford algorithm is  $O(ne)$ . The lines 1-2 takes  $O(n)$  time. If the matrix is stored in an adjacency list then lines 4-6 takes  $O(e)$  time. Hence, the lines 3-7 takes  $O(ne)$  time. Therefore, the several time of Bellman ford algorithm is  $O(ne)$ .



**Stepwise working procedure of Bellman Ford Algorithm**

## Assembly Line Scheduling



Assembly lines for automobile manufacturing factory

An automobile company has two assembly lines as shown in Fig.7.13. The automobile chassis enters an assembly line and goes through all stations of the assembly line and complete auto exists at the end of the assembly line. Each assembly line has  $n$  stations and the  $j^{\text{th}}$  station of  $i^{\text{th}}$  line is denoted as  $S_{ij}$ . The time required at station  $S_{ij}$  is  $a_{ij}$  and the time required to travel from one station to the next station is negligible. Normally, the chassis enters in a line goes through all the stations of the same line. However, in overload situations, the factory manager has the flexibility to switch partially completed auto from a station of one line to the next station of other line. The time required to transfer the partially completed auto from station  $S_{ij}$  to the other line is  $t_{ij}$ ; where  $i=1,2$  and  $j=1,2,\dots,n-1$ . Now the objective to choose some station from line 1 and some station from line 2 so that the total time to manufacture an auto can be minimized.

If there are many stations in the assembly lines the brute force search takes much time to determine the stations through which the auto can be assembled. There are  $2^n$  possible ways to choose stations from the assembly line. Thus determining the fastest way for assembling the auto takes  $O(2^n)$  time, which is infeasible when  $n$  is large. This problem can be efficiently solved by dynamic programming technique.

The first step of dynamic programming technique is to characterize the structure of an optimal solution. Since, there are 2 assembly lines with stations in each line, the computation of time to move to the 1<sup>st</sup> station of any assembly line is straight forward. However, there are two choices for  $j=2,3,\dots,n$  in each assembly line. First, the chassis may come from station  $S_{1,j-1}$  and then directly move to  $S_{1,j}$  since the time to move from one station to the next station in the same assembly line is negligible. The second choice is the chassis can come from station  $S_{2,j-1}$  and then be transferred to station  $S_{1,j}$  with a transfer time  $t_{2,j-1}$ . Let the fastest way through station  $S_{1,j}$  is through station  $S_{1,j-1}$ . Then there must be a fastest way through from the starting point through station  $S_{1,j-1}$ . Similarly, if there is a fastest way through station  $S_{2,j-1}$  then the chassis must have taken a fastest way from the starting point through station  $S_{2,j-1}$ .

Thus the optimal solution to the problem can be found by solving optimal solution of the sub-problems that in the fastest way to either  $S_{1,j-1}$  or  $S_{2,j-1}$ . This is referred as optimal structure of assembly line scheduling problem.

If we find the fastest way to solve assembly line scheduling problem through station  $j-1$  on either line 1 or line 2. Thus the fastest way through station  $S_{1,j}$  is either

- the fastest way to  $S_{1,j-1}$  and then directly through station  $S_{1,j}$ .
- the fastest way to  $S_{2,j-1}$  and a transfer from line 2 to line 1 and then through station  $S_{1,j}$ . Similarly, the fastest way through station  $S_{2,j}$  is
- the fastest way to  $S_{2,j-1}$  and then directly through station  $S_{2,j}$ .
- the fastest way to  $S_{1,j-1}$ , a transfer time from station 1.

Let  $f_i[j]$  be the fastest possible time to get a chassis from the starting point through station  $S_{i,j}$  of the assembly line  $i$ . Then chassis directly goes to the first station of each line.

$$f_1[1] = e_1 + a_{1,1}$$

$$f_2[1] = e_2 + a_{2,1}$$

Now, we can compute  $f_i[j]$ ,  $2 < j \leq n$  and  $1 \leq i \leq 2$  as

$$f_1[j] = \min\{f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}\}$$

$$f_2[j] = \min\{f_2[j-1] + a_{2,j}, f_1[j-1] + t_{1,j-1} + a_{2,j}\}$$

If the chassis goes all the way through station  $n$  either line 1 or line 2 and then exits, we have

$$f_1[n] = \begin{cases} e_1 + a_{1,1} & \text{if } n = 1 \\ \min\{f_1[n-1] + a_{1,n}, f_2[n-1] + t_{2,n-1} + a_{1,n}\} & \text{Otherwise} \end{cases}$$

$$f_2[n] = \begin{cases} e_2 + a_{2,1} & \text{if } n = 1 \\ \min\{f_2[n-1] + a_{2,n}, f_1[n-1] + t_{1,n-1} + a_{2,n}\} & \text{Otherwise} \end{cases}$$

Let the fastest time to get the chassis all the way through the factory is denoted by  $f^*$ . Then

$$f^* = \min\{f_1[n] + x_1, f_2[n] + x_2\}$$

Now, we can compute the stations through which the chassis must move to deliver the end product at minimum time. This can be calculated with a backward approach. Let  $l^*$  denote the line whose station  $n$  is used in a fastest way through the entire process. If  $f_1[n] + x_1 < f_2[n] + x_2$  then  $l^* = 1$  else  $l^* = 2$ . Then let us denote  $l_i[j]$  be the line number 1 or 2 whose station  $j-1$  is used in a fastest way through station  $S_{i,j}$ .  $l_i[1]$  is not required to be calculated since there is no station proceeding to station  $S_{i,1}$ .

If  $f_1[j-1] + a_{1,j} \leq f_2[j-1] + t_{2,j-1} + a_{1,j}$  then  $l_1[j] = 1$ . Otherwise,  $l_1[j] = 2$ . Similarly, if  $f_2[j-1] + a_{2,j} \leq f_1[j-1] + t_{1,j-1} + a_{2,j}$  then  $l_2[j] = 2$ . Otherwise,  $l_2[j] = 1$ . The algorithm for assembly line scheduling

problem is presented in Algorithm 7.7.

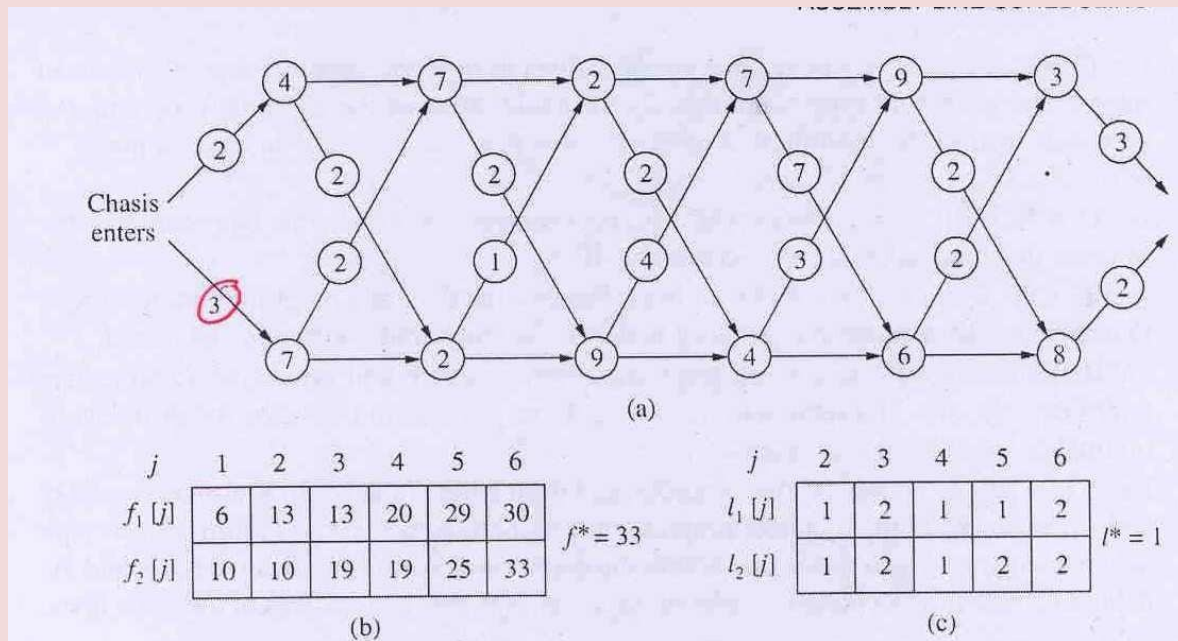


Illustration of assembly line scheduling procedure.

$$f_1[1] = e_1 + a_{1,1} = 2 + 4 = 6 \quad f_2[1] = e_2 + a_{2,1} = 3 + 7 = 10$$

$$f_1[2] = \min\{f_1[1] + a_{1,2}, f_2[1] + t_{2,1} + a_{1,2}\}$$

$$= \min\{6 + 7, 10 + 2 + 7\} = 13$$

$$f_2[2] = \min\{f_2[1] + a_{2,2}, f_1[1] + t_{1,1} + a_{2,2}\}$$

$$= \min\{10 + 2, 6 + 2 + 2\} = 10$$

The time required at all the stations of both the assembly lines are shown above. The minimum time required to assemble an auto is 33. The assembly line through which the complete auto is assembled is shown above.

## MODULE – III

### Disjoint Set Data Structure

In computing, a **disjoint-set data structure**, also called a **union–find data structure** or **merge–find set**, is a data structure that keeps track of a set of elements partitioned into a number

of disjoint (non-overlapping) subsets.

It supports the following useful operations:

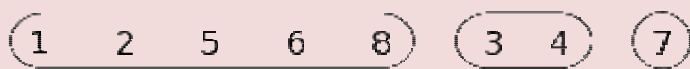
- *Find*: Determine which subset a particular element is in. *Find* typically returns an item from this set that serves as its "representative"; by comparing the result of two *Find* operations, one can determine whether two elements are in the same subset.
- *Union*: Join two subsets into a single subset.
- *Make Set*, which makes a set containing only a given element (a singleton), is generally trivial. With these three operations, many practical partitioning problems can be solved.

In order to define these operations more precisely, some way of representing the sets is needed. One common approach is to select a fixed element of each set, called its *representative*, to represent the set as a whole. Then, *Find*( $x$ ) returns the representative of the set that  $x$  belongs to, and *Union* takes two set representatives as its arguments.

Example :



*Make Set* creates 8 singletons.



After some operations of *Union*, some sets are grouped together.

### Applications :

- partitioning of a set
  - Boost Graph Library to implement its Incremental Connected Components functionality.
  - implementing Kruskal's algorithm to find the minimum spanning tree of a graph.
- Determine the connected components of an undirected graph.

### CONNECTED-COMPONENTS(G)

1. **for** each vertex  $v \in V[G]$
2. **do** MAKE-SET( $v$ )
3. **for** each edge  $(u,v) \in E[G]$
4. **doif** FIND-SET( $u$ )  $\neq$  FIND-SET( $v$ )
5. **then** UNION( $u,v$ )

SAME-COMPONENT( $u,v$ )

1. **if** FIND-SET( $u$ )=FIND-SET( $v$ )
2. **thenreturn** TRUE
3. **elsereturn** FALSE



## Disjoint Set Operations, Linked list Representation

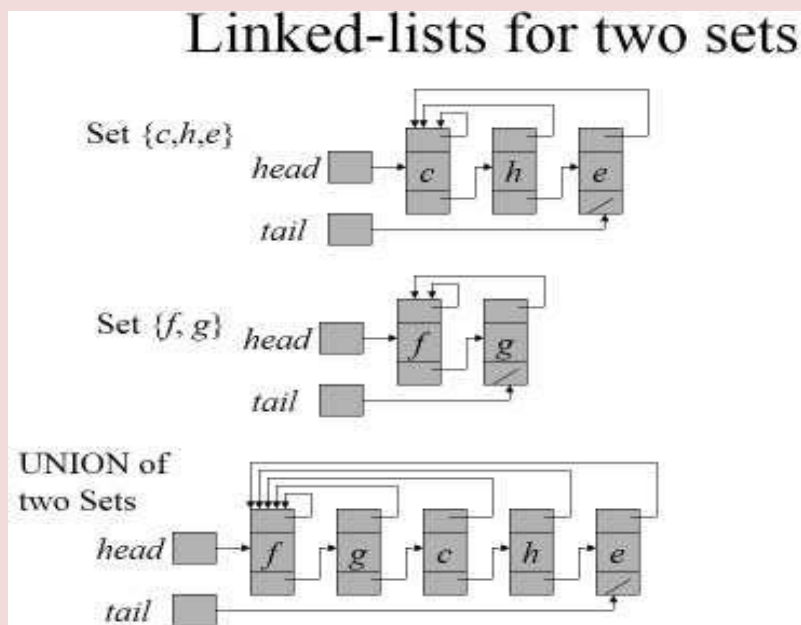
- A disjoint-set is a collection  $=\{S_1, S_2, \dots, S_k\}$  of distinct dynamic sets.
- Each set is identified by a member of the set, called *representative*.

### Disjoint set operations

- MAKE-SET( $x$ ): create a new set with only  $x$ . assume  $x$  is not already in someother set.
- UNION( $x,y$ ): combine the two sets containing  $x$  and  $y$  into one new set. A newrepresentative is selected.
- FIND-SET( $x$ ): return the representative of the set containing  $x$ .

### Linked list Representation

- Each set as a linked-list, with head and tail, and each node contains value, next nodepointer and back-to-representative pointer.
- Example:
- MAKE-SET costs  $O(1)$ : just create a single element list.
- FIND-SET costs  $O(1)$ : just return back-to-representative pointer.



## UNION Implementation

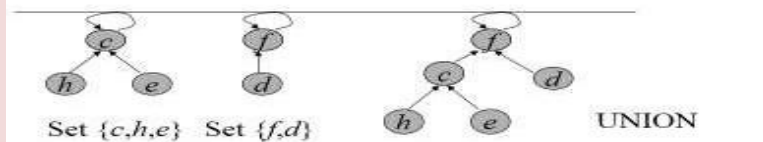
- A simple implementation:  $\text{UNION}(x,y)$  just appends  $x$  to the end of  $y$ , updates all back-to-representative pointers in  $x$  to the head of  $y$ .
- Each UNION takes time linear in the  $x$ 's length.
- Suppose  $n$  MAKE-SET( $x_i$ ) operations ( $O(1)$  each) followed by  $n-1$  UNION
  - $\text{UNION}(x_1, x_2), O(1),$
  - $\text{UNION}(x_2, x_3), O(2),$
  - .....
  - $\text{UNION}(x_{n-1}, x_n), O(n-1)$
- The UNIONS cost  $1+2+\dots+n-1=(n^2)$

So  $2n-1$  operations cost  $(n^2)$ , average  $(n)$  each

## Disjoint Forests

### Disjoint-set Implementation: Forests

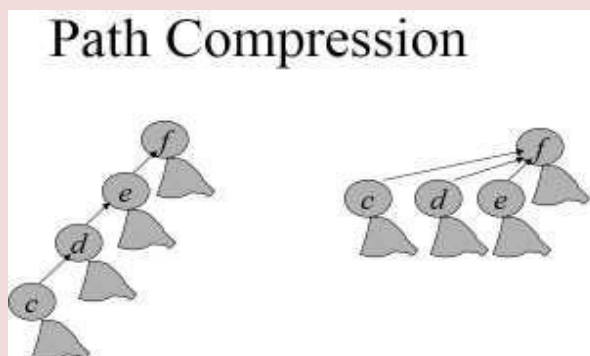
- Rooted trees, each tree is a set, root is the representative. Each node points to its parent. Root points to itself.



- Three operations
  - MAKE-SET( $x$ ): create a tree containing  $x$ .  $O(1)$
  - FIND-SET( $x$ ): follow the chain of parent pointers until to the root.  $O(\text{height of } x\text{'s tree})$
  - UNION( $x,y$ ): let the root of one tree point to the root of the other.  $O(1)$
- It is possible that  $n-1$  UNIONS results in a tree of height  $n-1$ . (just a linear chain of  $n$  nodes).
- So  $n$  FIND-SET operations will cost  $O(n^2)$ .

## Union by Rank & Path Compression

- **Union by Rank:** Each node is associated with a rank, which is the upper bound on the height of the node (i.e., the height of subtree rooted at the node), then when UNION, let the root with smaller rank point to the root with larger rank.
- **Path Compression:** used in FIND-SET( $x$ ) operation, make each node in the path from  $x$  to the root directly point to the root. Thus reduce the tree height.



## Algorithm for Disjoint-Set Forest

<b>MAKE-SET(<math>x</math>)</b> 1. $p[x] \leftarrow x$ 2. $rank[x] \leftarrow 0$	<b>UNION(<math>x, y</math>)</b> 1. LINK(FIND-SET( $x$ ), FIND-SET( $y$ ))	<b>FIND-SET(<math>x</math>)</b> 1. <b>if</b> $x \neq p[x]$ 2. <b>then</b> $p[x] \leftarrow$ FIND-SET( $p[x]$ ) 3. <b>return</b> $p[x]$
	<b>LINK(<math>x, y</math>)</b> 1. <b>if</b> $rank[x] > rank[y]$ 2. <b>then</b> $p[y] \leftarrow x$ 3. <b>else</b> $p[x] \leftarrow y$ 4. <b>if</b> $rank[x] = rank[y]$ 5. <b>then</b> $rank[y]++$	

Worst case running time for  $m$  MAKE-SET, UNION, FIND-SET operations is:  $O(m\alpha(n))$  where  $\alpha(n) \leq 4$ . So nearly linear in  $m$ .

## Graph Algorithm - BFS and DFS

In graph theory, **breadth-first search (BFS)** is a strategy for searching in a graph when search is limited to essentially two operations:

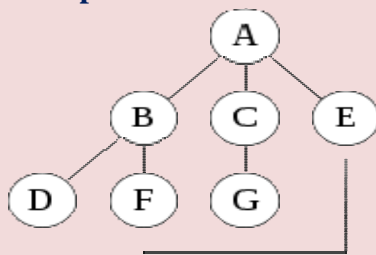
- (a) visit and inspect a node of a graph;
- (b) gain access to visit the nodes that neighbor the currently visited node.

- The BFS begins at a root node and inspects all the neighboring nodes.
- Then for each of those neighbor nodes in turn, it inspects their neighbor nodes which were unvisited, and so on.
- Compare BFS with the equivalent, but more memory-efficient.

### Historical Background

- BFS was invented in the late 1950s by E. F. Moore, who used to find the shortest path out of a maze,
- discovered independently by C. Y. Lee as a wire routing algorithm (published 1961).

### Example



A BFS search will visit the nodes in the following order: A, B, C, E, D, F, G

### BFS Algorithm

The algorithm uses a queue data structure to store intermediate results as it traverses the graph, as follows:

1. Enqueue the root node
2. Dequeue a node and examine it
  - If the element sought is found in this node, quit the search and return a result.
  - Otherwise enqueue any successors (the direct child nodes) that have not yet been discovered.
3. If the queue is empty, every node on the graph has been examined – quit the search and return "not found".
4. If the queue is not empty, repeat from Step 2.

## Applications

Breadth-first search can be used to solve many problems in graph theory, for example:

- Copying Collection, Cheney's algorithm
- Finding the shortest path between two nodes  $u$  and  $v$  (with path length measured by number of edges)
- Testing a graph for bipartiteness
- (Reverse) Cuthill–McKee mesh numbering
- Ford–Fulkerson method for computing the maximum flow in a flow network
- Serialization/Deserialization of a binary tree vs serialization in sorted order, allows the tree to be re-constructed in an efficient manner.

## Pseudo Code

```
1  procedure BFS( $G, v$ ) is
2      create a queue  $Q$ 
3      create a set  $V$ 
4      add  $v$  to  $V$ 
5      enqueue  $v$  onto  $Q$ 
6      while  $Q$  is not empty loop
7           $t \leftarrow Q.dequeue()$ 
8          if  $t$  is what we are looking for then
9              return  $t$ 
10             end if
11             for all edges  $e$  in  $G.adjacentEdges(t)$  loop
12                  $u \leftarrow G.adjacentVertex(t, e)$ 
13                 if  $u$  is not in  $V$  then
14                     add  $u$  to  $V$ 
15                     enqueue  $u$  onto  $Q$ 
16                 end if
17             end loop
18         end loop
19         return none
20 end BFS
```

**Input:** A graph  $G$  and a root  $v$  of  $G$

## Time and space complexity

The time complexity can be expressed as  $O(|V| + |E|)$  since every vertex and every edge will be explored in the worst case. Note:  $O(|E|)$  may vary between  $O(1)$  and  $O(|V|^2)$ , depending on how sparse the input graph is.

When the number of vertices in the graph is known ahead of time, and additional data structures are used to determine which vertices have already been added to the queue, the space complexity can be expressed as  $O(|V|)$  where  $|V|$  is the cardinality of the set of vertices. If the graph is

represented by an Adjacency list it occupies  $O(|V|)$  space in memory, while an Adjacency matrix representation occupies  $O(|V|^2)$ .

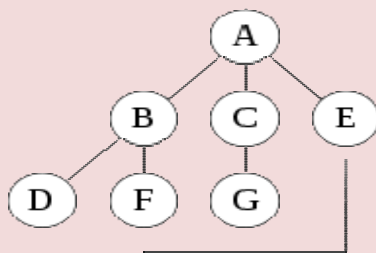
**Depth-first search (DFS)** is an algorithm for traversing or searching tree or graph data structures.

One starts at the root (selecting some arbitrary node as the root in the case of a graph) and explores as far as possible along each branch before backtracking.

## Historical Background

A version of depth-first search was investigated in the 19th century by French mathematician Charles Pierre Trémaux.

## Example



A DFS search will visit the nodes in the following order: A, B, D, F, E, C, G

## Pseudo Code

**Input:** A graph  $G$  and a vertex  $v$  of  $G$

**Output:** All vertices reachable from  $v$  labeled as discovered

## A recursive implementation of DFS

```
1 procedure DFS( $G, v$ ):
2   label  $v$  as discovered
3   for all edges from  $v$  to  $w$  in  $G$ .adjacentEdges( $v$ ) do
4     if vertex  $w$  is not labeled as discovered then
5       recursively call DFS( $G, w$ )
```

## A non-recursive implementation of DFS

```
1 procedure DFS-iterative( $G, v$ ):
2   let  $S$  be a stack
3    $S.push(v)$ 
4   while  $S$  is not empty
5      $v \leftarrow S.pop()$ 
6     if  $v$  is not labeled as discovered:
7       label  $v$  as discovered
8       for all edges from  $v$  to  $w$  in  $G.adjacentEdges(v)$  do
9          $S.push(w)$ 
```

## Applications

- Finding connected components.
- Topological sorting.
- Finding the bridges of a graph.
- Generating words in order to plot the Limit Set of a Group.
- Finding strongly connected components.
- Planarity testing
- Solving puzzles with only one solution, such as mazes. (DFS can be adapted to find all solutions to a maze by only including nodes on the current path in the visited set.)
- Maze generation may use a randomized depth-first search.
- Finding bi-connectivity in graphs.

## DESIGN AND ANALYSIS OF ALGORITHMS QUESTION BANK UNIQUE PAPER

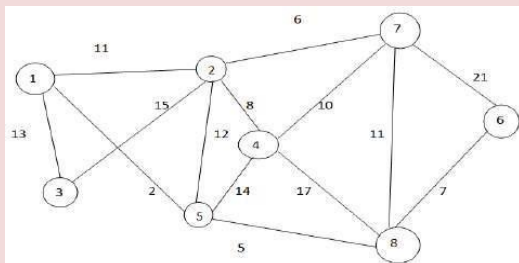
1. Devise an algorithm that sorts a collection of  $n \geq 1$  elements of arbitrary type.
2. State the best, average and worst case complexities of binary search for successful and unsuccessful search.
3. State the principle of optimality. Find two problems for which the principle does not hold.
4. Determine the frequency counts for all statements in the following algorithm segment.

```
i:=1;  
while(i≤n) do  
{  
x:=x+1;  
i:=i+1;  
}
```

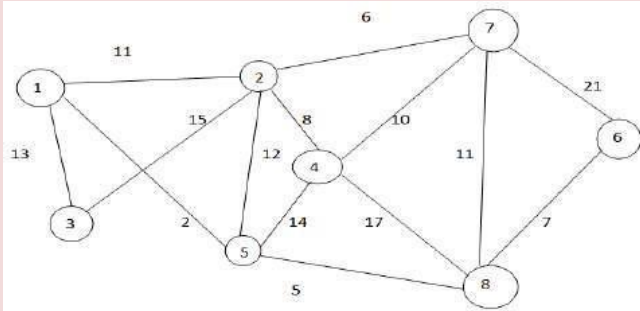
5. Solve the recurrence relation using substitution  
method  $T(n) = \begin{cases} T(1) & n=1 \\ aT(n/b)+f(n) & n>1 \end{cases}$ , where  $a=5, b=4$ , and  $f(n)=cn^2$ .
6. Apply quick sort algorithm to sort the list. E, X, A, M, P, L, E in alphabetical order.
7. Analyze the best, average and worst case complexity of quick sort.
8. Compare BFS and DFS algorithm with an example graph and denote its time complexities.
9. Derive time complexity of job sequencing with deadlines. Obtain the optimal solution when  $n=5$ ,  $(p_1, p_2, \dots) = (20, 15, 10, 5, 1)$  and  $(d_1, d_2, \dots) = (2, 2, 1, 3, 3)$ .
10. Describe about reliability design with an example.
11. Obtain the solution to knapsack problem by Dynamic Programming  
method  $n=6, (p_1, p_2, \dots, p_6) = (w_1, w_2, \dots, w_6) = (100, 50, 20, 10, 7, 3)$  and  $m=165$ .
12. Explain how backtracking is used for solving n- queens problem. Show the state space tree.
13. Describe the algorithm for Hamiltonian cycles and determine the order of magnitude of the worst-case computing time for the backtracking procedure that finds all Hamiltonian cycles.
14. Describe the Travelling sales person problem and discuss how to solve it using dynamic programming.



15. What are the four distinct areas of study of algorithm?
16. Is quick sort a stable sorting method? Justify.
17. Can we say that the time for Merge Sort is  $\Theta(n \log n)$ . What is its worst and best time of procedure for Merge Sort.
18. Use an algorithm for greedy strategies for the knapsack to find an optimal solution to the knapsack instance  $n=7, m=15, (p_1, p_2, \dots, p_7)=(10, 5, 15, 7, 6, 18, 3)$ , and  $(w_1, w_2, \dots, w_7)=(2, 3, 5, 7, 1, 4, 1)$ .
19. Apply greedy algorithm to generate single-source shortest path with an example graph. Mention its time complexity.
20. Write about three popular methods to arrive at amortized costs for operations with.
21. What is stable sorting method? Is merge sort a stable sorting method? Justify.
22. Define spanning tree. Compute a minimum cost spanning tree for the graph of figure using prim's algorithm.



23. What is knapsack problem? State knapsack problem formally.
24. Distinguish Greedy method and Dynamic Programming.
25. Define spanning tree. Compute a minimum cost spanning tree for the graph of figure using kruskal's algorithm.



26. What is back tracking? Where Back tracking is used to solve the problem.
27. What is the difference between 0/1 Knapsack problem and fractional Knapsack problem.
28. Explain the Quick Sort algorithm with an example and also draw the tree structure of the recursive calls made.
29. Explain the Merge Sort algorithm with an e.g. and also draw the tree structure of the recursive calls made.
30. Give the Binary search algorithm and analyze the efficiency.
31. Write an algorithm of BFS? Also give an example
32. Write an algorithm of DFS? Also give an example.
33. Explain the various criteria used for analyzing algorithms.
34. List the properties of various asymptotic notations.
35. What is the average case complexity of linear search algorithm?
36. Differentiate dynamic programming and divide and conquer.
37. State the time complexity of bubble sort algorithm.
38. Apply backtracking technique to solve the following instance of the subset sum problem  $S = \{1,3,4,5\}$  and  $d=11$  16
39. Explain subset-sum problem and discuss the possible solution strategies using backtracking.
40. What is tree edge and cross edge?
41. Define back edge and tree edge.
42. Explain graph coloring.

